IDENTITIES IN THE LATTICES OF SEMIGROUP VARIETIES

A Summary of a Dissertation
submitted to the St. Petersburg State University
in Fulfillment of the Requirements for the Degree
of Doctor of Physical and Mathematical Sciences*

by M. V. VOLKOV

1 A general description of the dissertation

The lattice of all group varieties is modular while the lattice of all semigroup varieties satisfies no non-trivial lattice identity. Comparing of these fundamental facts naturally leads to the problem of describing semigroup varieties with a *modular subvariety lattice*; in the sequel, for brevity, we call such varieties *modular*. In explicit form this problem was first mentioned more then 20 years ago in a well-known survey paper by Evans [1], although in reality it had already become an object of intensive research in the mid-sixties (starting with a dissertation by Schwabauer [8]). By now questions which are more or less closely connected with modular varieties have taken an important places in the theory of semigroup varieties, as evidenced, inter alia, by the fact that about two hundred papers have been devoted to its various aspects. In these articles the modularity appears in two ways: first as a stimulus for working out and perfecting research methods for varietal lattices (in particular, in the direction of solving Evans' problem or the similar problem of describing semigroup varieties with a *distributive subvariety lattice* posed in the mid-seventies by L.N. Shevrin [10]), and second as a tool for such research that is especially handy in looking for "good" decompositions of the lattices into subdirect and direct products. Many substantial results have been obtained which have essentially clarified the structure of certain important parts of the lattice of all semigroup varieties (in particular, the lattice of *completely regular semigroup* varieties was deeply investigated

*The Dissertation was defended on May 25, 1994.

[7] and proved to be modular [4], but Evans' problem per se remained open up to now.

The aim of the dissertation is to give an exhaustive characterization of semigroup varieties with modular subvariety lattices, i.e. a complete solution to the problem of Evans. We investigate also the structure of lattices of varieties consisting of semigroups being "far from groups", in particular *nilsemigroups*.

The general plan of our investigation is the following. First of all, we have essentially generalized some known and developed some new methods for constructing semigroup varieties with a non-modular subvariety lattice. This allowed to discover new necessary conditions for the modularity of the lattices of semigroup varieties. Using these conditions we were able to subdivide the initial problem into several more concrete parts requiring some specific technique. Namely, roughly speaking, describing modular varieties have been reduced to the cases when the variety consists of semigroups close to either completely regular or nilpotent. To handle the former case we need to expand to an essentially larger class of semigroups the elaborated technique developed while studying completely regular varieties. We have managed to generalize most of the previously known modularity conditions and to prove the modularity of the lattice of varieties of semigroups with completely regular square. In the opposite, the analysis of the "almost nilpotent" case has led to an unexplored country. Here we have developed an absolutely new approach based on a deep analogy with the ring case and connected with parametrizing nilpotent semigroup varieties by means of (non-linear) representations of symmetric groups. This approach allowed to classify all modular varieties of nilsemigroups.

All the main results of the dissertation are new as well as the most auxiliary results. They will help in answering several related questions about the lattices of semigroup varieties, in particular, in developing the theory of subdirect decompositions of the lattices and in lattice characterizations of some properties of varieties. The theory of the lattices of nilsemigroup varieties developed on the dissertation can be used in essentially more general situations than the semigroup one. Some of the phenomena discovered in the dissertation — e.g,. a paradoxical character of the interconnection between the permutability of fully invariant congruences on free semigroups and identities in the lattices of semigroup varieties — are also of universal-algebraic nature and their importance goes out the framework of the theory of semigroups.

The results of the dissertation were reported at the numerous semigroup and universal-algebraic conferences in Russia, Germany, Austria, Italy, the USA, Australia, Poland, Czech Republic and Slovakia. They have appeared in 23 publications by the author: journal papers [12,28,32,38–42], contributions to books [4–6], published contributions to conferences [8,18–20,24,27,29,31–34,36] (numbers in brackets refer to the enclosed list of author's publications). The dissertation consists of an introduction, 5 chapters and a conclusion. Its volume (without the bibliography) is 300 pages; the bibliography contains 227 items.

2 A survey of the results of the dissertation

The main result of the dissertation is a complete description of semigroup varieties with a modular subvariety lattice. We characterize the varieties from various points of view: first by exhibiting a complete list of identity systems such that their validity in a variety is equivalent to the modularity of its subvariety lattice; second by indicating structure properties of a variety and semigroups it contains which are equivalent to the modularity; third by listing all *just-non-modular varieties*, i.e. varieties minimal with respect to the property "to have a non-modular subvariety lattice". Using the terminology introduced by L.N.Shevrin in the survey [9], we can say that we give *equational, structural* and *indicator* characterizations of varieties with a modular subvariety lattice respectively.

MAIN THEOREM (equational version). *The lattice of all subvarieties of a semigroup variety $\mathcal{V}$ is modular if and only if $\mathcal{V}$ satisfies:*

- *the identity*
$$xy = (xy)^n \quad (n > 1); \tag{1}$$
- *or one of the identity systems*
$$xy = x^n y,\ (xy)^n = xy^n,\ xyzt = xyx^{n-1}zt \quad (n > 1); \tag{2}$$
$$xy = xy^n,\ (xy)^n = x^n y,\ xyzt = xyt^{n-1}zt \quad (n > 1); \tag{3}$$
- *or the identity*
$$x_1x_2x_3x_4 = x_{1\vartheta}x_{2\vartheta}x_{3\vartheta}x_{4\vartheta}, \tag{4}$$
where ϑ is one of the permutations $\langle 123\rangle, \langle 124\rangle, \langle 134\rangle, \langle 234\rangle, \langle 12\rangle\langle 34\rangle, \langle 13\rangle\langle 24\rangle, \langle 14\rangle\langle 32\rangle$, together with one of the following 17 systems of identities:
$$x^2y = xyx = yx^2 = x^m y \quad (m > 2); \tag{5}$$
$$x^2y = xyx = yx^2,\ x^3yz = xy^3z,\ x^6 = x^7; \tag{6}$$
$$x^2y = xyx = yx^2,\ x^2y^2z = xy^2z^2; \tag{7}$$
$$x^2y = xyx = yx^2,\ x^3yz = xy^2z^2; \tag{8}$$

$$x^2y = xyx = xy^2,\ x^4y = yx^4; \tag{9}$$

$$x^2y = yxy = xy^2,\ x^4y = yx^4; \tag{10}$$

$$x^2y = xyx,\ yx^2 = xy^2; \tag{11}$$

$$x^2y = y^2x,\ yx^2 = xyx; \tag{12}$$

$$x^2y = yxy,\ yx^2 = xy^2; \tag{13}$$

$$x^2y = y^2x,\ yx^2 = yxy; \tag{14}$$

$$x^2y = y^2x,\ xyx = x^2yx; \tag{15}$$

$$xyx = xyx^2,\ yx^2 = xy^2; \tag{16}$$

$$x^2y = x^3y,\ xyx = yxy; \tag{17}$$

$$yx^2 = yx^3,\ xyx = yxy; \tag{18}$$

$$x^2y = yx^2 = yxy; \tag{19}$$

$$x^2y = yx^2,\ xyx = yxy; \tag{20}$$

$$x^2y = xy^2,\ xyx = yxy; \tag{21}$$

- *or one of the identity systems*

$$x^2y = x^3y,\ yx^2 = xy^2,\ x_1x_2x_3x_4 = x_{1\zeta}x_{2\zeta}x_{3\zeta}x_{4\zeta}; \tag{22}$$

$$x^2y = y^2x,\ yx^2 = yx^3,\ x_1x_2x_3x_4 = x_{1\zeta}x_{2\zeta}x_{3\zeta}x_{4\zeta}, \tag{23}$$

where ζ *is one of the permutations* $\langle 123\rangle, \langle 124\rangle, \langle 134\rangle, \langle 234\rangle, \langle 12\rangle\langle 34\rangle$*;*

- *or one of the identity systems* (22) *or* (23), *where* $\zeta = \langle 13\rangle\langle 24\rangle$, *together with one of the identities*

$$xyxz = yxzx; \tag{24}$$

$$xyxz = yxyz; \tag{25}$$

$$yxzx = yzxz; \tag{26}$$

- *or one of the identity systems* (22) *or* (23), *where* $\zeta = \langle 14\rangle\langle 23\rangle$, *together with one of the identities*

$$xyzx = xyxz; \tag{27}$$

$$xyzx = yxyz; \tag{28}$$

$$xyzx = yzyx; \tag{29}$$

$$xyxz = yzyx; \tag{30}$$

$$xyzx = yxzy; \tag{31}$$

- *or one of the identity systems*

$$x^2y = y^2x, \ yx^2 = (yx)^2; \tag{32}$$

$$x^2y = (xy)^2, \ yx^2 = xy^2; \tag{33}$$

together with one of the identity systems

$$x_1x_2x_3x_4 = x_2x_1x_4x_3; \tag{34}$$

$$x_1x_2x_3x_4 = x_3x_4x_1x_2, \ xyxz = yxzx; \tag{35}$$

$$x_1x_2x_3x_4 = x_4x_3x_2x_1, \ xyzx = yxzy. \tag{36}$$

Thus, there exists a *countable* set of identity systems (more exactly, 10 series indexed by positive integers and 144 "sporadic" systems) such that each of these systems is *finite* (more exactly, consists of no more than 5 identities) and such that their validity in a variety is equivalent to the modularity of its subvariety lattice.

To formulate a description of modular varieties in a structure language, we need some notation and definitions. Let

- $\mathcal{T}$ be the trivial variety;
- $\mathcal{SL}$ be the variety of all commutative idempotent semigroups (semilattices);
- $\mathcal{C}$ be the variety generated by the semigroup $C = \{0, c, 1\}$, where 0 is zero, 1 is the identity element, and $c^2 = 0$;
- $\mathcal{P}$ be the variety generated by the semigroup

$$P = \langle p, e|\ e^2 = e, \ ep = p, \ pe = 0\rangle;$$

- $\mathcal{Q}$ be the variety generated by the semigroup

$$Q = \langle q, e, f|\ ef = f, \ fe = e, \ q^2 = fq = qe = 0\rangle.$$

Given a variety $\mathcal{X}$, we denote by $\overleftarrow{\mathcal{X}}$ the dual variety. By $\mathcal{X} \bigvee \mathcal{Y}$ we denote the lattice *join* of the varieties $\mathcal{X}$ and $\mathcal{Y}$, i.e. the least variety containing both $\mathcal{X}$ and $\mathcal{Y}$. Recall that a semigroup is said to be *completely regular* if it coincides with the set-theoretical union of its subgroups and a semigroup with zero is called *a nilsemigroup* if some power of each of its elements equals zero. Semigroup satisfying the identity $xyz = xyxz$ ($xyz = xzyz$) are said to be *left self-distributive* (resp., *right self-distributive*).

MAIN THEOREM (structural version). *The lattice of all subvarieties of a semigroup variety $\mathcal{V}$ is modular if and only if $\mathcal{V}$ satisfies one of the following conditions:*

M1 *for every semigroup $S \in \mathcal{V}$, the semigroup $S^2 = \{st|\ s, t \in S\}$ is completely regular;*

M2 *$\mathcal{V} = \mathcal{D} \bigvee \mathcal{X}$, where the variety $\mathcal{D}$ consists of completely regular semigroups whose idempotents form a left self-distributive subsemigroup and $\mathcal{X}$ is one of the varieties $\mathcal{P}$ or $\mathcal{Q}$;*

M2′ *$\mathcal{V} = \mathcal{D} \bigvee \mathcal{X}$, where the variety $\mathcal{D}$ consists of completely regular semigroups whose idempotents form a right self-distributive subsemigroup and $\mathcal{X}$ is one of the varieties $\overleftarrow{\mathcal{P}}$ or $\overleftarrow{\mathcal{Q}}$;*

M3 *$\mathcal{V} = \mathcal{N} \bigvee \mathcal{G} \bigvee \mathcal{X}$, where the variety $\mathcal{G}$ consists of abelian groups, $\mathcal{X}$ is one of the varieties $\mathcal{T}$, $\mathcal{SL}$ or $\mathcal{C}$, and the variety $\mathcal{N}$ satisfies the identities $x^2y = ztz = yx^2$ and an identity of the kind* (4)*, where ϑ is one of the permutations* $\langle 123\rangle, \langle 124\rangle, \langle 134\rangle, \langle 234\rangle, \langle 12\rangle\langle 34\rangle, \langle 13\rangle\langle 24\rangle, \langle 14\rangle\langle 32\rangle$*;*

M4 *$\mathcal{V} = \mathcal{M} \bigvee \mathcal{X}$, where $\mathcal{X}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, and the variety $\mathcal{M}$ consists of nilsemigroups and satisfies:*

- *an identity of the kind* (4) *where ϑ is one of the permutations* $\langle 123\rangle, \langle 124\rangle, \langle 134\rangle, \langle 234\rangle, \langle 12\rangle\langle 34\rangle, \langle 13\rangle\langle 24\rangle, \langle 14\rangle\langle 32\rangle$*, together with one of the identity systems* (6)–(21)*;*
- *or one of the identity systems of the kind* (22) *or* (23)*, where ζ is one of the permutations* $\langle 123\rangle, \langle 124\rangle, \langle 134\rangle, \langle 234\rangle, \langle 12\rangle\langle 34\rangle$*;*
- *or one of the identity systems* (22) *or* (23)*, where $\zeta = \langle 13\rangle\langle 24\rangle$, together with one of the identities* (24)–(26)*;*
- *or one of the identity systems* (22) *or* (23)*, where $\zeta = \langle 14\rangle\langle 23\rangle$, together with one of the identities* (27)–(31)*;*
- *or one of the identity systems* (32) *or* (33) *together with one of the identity systems* (34)–(36).

Although the characterizations above are quite complete, it is worth adding to them the indicator characterization which naturally arises when the main theorem being proved. Our experience teaches that the indicator characterization is especially handy when it is necessary to check whether a concrete variety satisfies the desired property.

Let varΣ denote the variety given by the identity system Σ. To formulate the indicator characterization of varieties with a modular subvariety lattice, we have to introduce notation for few additional varieties:

$$\mathcal{CZM} = \mathrm{var}\{xyz = y^2,\ xy = yx\},$$
$$\mathcal{X}_0 = \mathrm{var}\{xyzt = y^3,\ xy = yx,\ xyx = yxy\},$$
$$\mathcal{Y}_{\langle 12\rangle} = \mathrm{var}\{xyzt = y^2z,\ x_1x_2x_3 = x_2x_1x_3,\ xyx = yxy\},$$
$$\mathcal{Y}_{\langle 13\rangle} = \mathrm{var}\{xyzt = y^2,\ x_1x_2x_3 = x_3x_2x_1,\ xyx = yxy\},$$
$$\mathcal{Y}_{\langle 23\rangle} = \mathrm{var}\{xyzt = yz^2,\ x_1x_2x_3 = x_1x_3x_2,\ xyx = yxy\},$$
$$\mathcal{Z}_{\langle 12\rangle} = \mathrm{var}\{xyzt = yzy,\ x_1x_2x_3 = x_2x_1x_3,\ x^2y = y^2x\},$$
$$\mathcal{Z}_{\langle 13\rangle} = \mathrm{var}\{xyzt = yzy,\ x_1x_2x_3 = x_3x_2x_1,\ x^2y = xy^2\},$$
$$\mathcal{Z}_{\langle 23\rangle} = \mathrm{var}\{xyzt = yzy,\ x_1x_2x_3 = x_1x_3x_2,\ yx^2 = xy^2\},$$
$$\mathcal{LZ} = \mathrm{var}\{xy = x\},\quad \mathcal{RZ} = \mathrm{var}\{xy = y\},$$
$$\mathcal{LZM} = \mathrm{var}\{xyz = xy\},\quad \mathcal{RZM} = \mathrm{var}\{xyz = yz\},$$
$$\mathcal{LRB} = \mathrm{var}\{x^2 = x,\ xy = xyx\},\quad \mathcal{RRB} = \mathrm{var}\{x^2 = x,\ yx = xyx\},$$
$$\mathcal{LSNB} = \mathrm{var}\{x^2 = x,\ xyz = xyzxz\},$$
$$\mathcal{RSNB} = \mathrm{var}\{x^2 = x,\ xyz = xzxyz\},$$
$$\mathcal{A}_p = \mathrm{var}\{x^py = y,\ xy = yx\},$$
$$\mathcal{CSA}_p = \mathrm{var}\{(xy)^px = x,\ xyx^2 = x^2yx\}.$$

By $\mathbf{Sym}(n)$ we denote the group of all permutations of the set $\{1, 2, .., n\}$.

MAIN THEOREM (indicator version). *The lattice of all subvarieties of a semigroup variety $\mathcal{V}$ is modular if and only if $\mathcal{V}$ contains none of the following varieties:*

$$\mathcal{CZM}\bigvee\mathcal{LZ},\ \mathcal{CZM}\bigvee\mathcal{RZ},\ \mathcal{CZM}\bigvee\mathcal{P},\ \mathcal{CZM}\bigvee\overleftarrow{\mathcal{P}},\ \mathcal{CZM}\bigvee\mathcal{Y},$$

where $\mathcal{Y}$ runs over all just-non-abelian varieties of periodic groups;

$$\mathcal{X}_0\bigvee\mathcal{C},\ \mathcal{Y}_\tau\bigvee\mathcal{C},\ \mathcal{Z}_\tau\bigvee\mathcal{C},\ \mathcal{X}_0\bigvee\mathcal{A}_p, \mathcal{Y}_\tau\bigvee\mathcal{A}_p,\ \mathcal{Z}_\tau\bigvee\mathcal{A}_p,$$

where τ runs over all transpositions from $\mathbf{Sym}$*(3) and p runs over all prime numbers;*

$$\mathcal{P}\bigvee\mathcal{RZM},\ \overleftarrow{\mathcal{P}}\bigvee\mathcal{LZM},\ \mathcal{P}\bigvee\mathcal{RRB},\ \overleftarrow{\mathcal{P}}\bigvee\mathcal{LRB},$$

$$\mathcal{P}\bigvee\mathcal{LSNB},\ \overleftarrow{\mathcal{P}}\bigvee\mathcal{RSNB},\ \mathcal{P}\bigvee\mathcal{CSA}_p,\ \overleftarrow{\mathcal{P}}\bigvee\mathcal{CSA}_p,$$

where p runs over all prime numbers;

$$\mathrm{var}\{x^2 = xyx = x_1 \ldots x_5,\ x_1x_2x_3x_4 = x_{1\sigma}x_{2\sigma}x_{3\sigma}x_{4\sigma}\},$$

where σ runs over all transpositions from **Sym***(4);*

$$\mathrm{var}\{y^3 = xyzt,\ x_1x_2x_3 = x_{1\tau}x_{2\tau}x_{3\tau}\},$$

where τ runs over all transpositions from **Sym***(3);*

$$\mathrm{var}\{xyzt = z^2y\},\ \mathrm{var}\{xyzt = zyz\},\ \mathrm{var}\{xyzt = zy^2\};$$

$$\mathrm{var}\{xyzt = y^3,\ x^2y = yxy\},\ \mathrm{var}\{xyzt = y^3,\ xy^2 = xyx\},$$
$$\mathrm{var}\{xyzt = y^3,\ x^2y = xy^2\};$$

$$\mathrm{var}\{xyzt = y^3,\ x^2y = y^2x,\ xy^2 = yx^2,\ xyx = yxy\};$$

$$\mathrm{var}\{x_1 \dots x_5 = x^2,\ x_1x_2x_3x_4 = x_4x_3x_2x_1,\ xyxz = xzxy\},$$

$$\mathrm{var}\{x_1 \dots x_5 = x^2,\ x_1x_2x_3x_4 = x_4x_3x_2x_1,\ xyxz = yxyz\},$$

$$\mathrm{var}\{x_1 \dots x_5 = x^2,\ x_1x_2x_3x_4 = x_4x_3x_2x_1,\ xyxz = zyzx\};$$

$$\mathrm{var}\{x_1 \dots x_5 = x^2,\ x_1x_2x_3x_4 = x_3x_4x_1x_2\};$$

$$\mathrm{var}\{x_1 \dots x_6 = x^4 = x^3y^2,\ xy = yx\}.$$

The varieties listed in the last version of the main theorem are precisely exactly all just-non-modular varieties of semigroups. (We call a variety *just-non-modular* if its subvariety lattice is modular while each of its proper subvarieties is modular.) Thus, there exists one continual series of just-non-modular varieties (indexed by the just-non-abelian varieties of periodic groups, whose set is continual by [3]), 9 countable series (indexed by primes), and 38 "sporadic" varieties. Observe that there exist just-non-modular varieties that are not finitely based (compare that with the remark made after the equational version of the main theorem).

Other results of the dissertation group around the main theorem being either steps of its proof or its applications or its variations where the modularity is substituted by, say, distributivity (or another related condition). We briefly describe the distribution of the material in the dissertation.

Chapter 0 consists of section 0–6 and contains all necessary preliminaries. We note that many of the results presented in sections 2,3,5, and 6 are new and, although play a supporting role in this dissertation, often are of an independent interest.

Chapter 1 is devoted to studying the structure of the subvariety lattices of nilpotent semigroup varieties. First we prove in section 7 that studying identities in the lattices of varieties of nilsemigroup reduces to the case of nilpotent semigroup varieties. In sections 8 and 9 we show that the subvariety lattice of a nilpotent semigroup variety can be reconstructed from some its intervals. Finally, in the key section 10 the structure of the latter intervals is completely determined in terms of a certain action of a symmetric group on a set. To formulate the central result of Chapter 1, Theorem 10.1, we need some definitions.

Let M be a set and $\varphi : G \to \mathbf{Sym}(M)$ be a homomorphism of a group G into the group of all permutations of the set M. In such the situation, we consider M as a unary algebra with the set G of operations (the operation $g \in G$ is defined by the rule $mg = m\varphi(g)$) and call it a *G-set*. The notions of a subalgebra and a congruence of a G-set have a standard meaning. We call a pair (ρ, N), where ρ is a congruence of a G-set M and N is a subalgebra of M, *coherent* if N is a ρ-class. Let us agree that the empty set is both a subalgebra of the G-set M and a class of any of its congruences; then the pair $(\rho, \emptyset)$ is coherent for any congruence ρ on M. Let CP(M) denote the set of all coherent pairs over M. It is easy to see that CP(M) becomes a complete lattice under the natural ordering: $(\rho, N) \leq (\mu, L)$ if and only if $\rho \subseteq \mu$ and $N \subseteq L$.

Let $\mathcal{V}$ be a semigroup variety and $\alpha_{\mathcal{V}}$ be the fully invariant congruence on the free semigroup F corresponding to $\mathcal{V}$. We call a variety $\mathcal{V}$ *homogeneous* if all words from an arbitrary $\alpha_{\mathcal{V}}$-class A have equal length provided $A \neq 0$ in the quotient semigroup $F/\alpha_{\mathcal{V}}$. For any pair (k, m) of positive integers such that $m \leq k$, consider in the free semigroup F the set $F[k, m; \mathcal{V}]$ of all words of length k which depend precisely on the letters from the set $\{x_1, \ldots, x_m\}$ and do not belong to the zero $\alpha_{\mathcal{V}}$-class. A subset $W \subseteq F[k, m; \mathcal{V}]$ is said to be a $\mathcal{V}$-cross section if:

1) different words from W belong to different $\alpha_{\mathcal{V}}$-classes;

2) for each $u \in F[k, m; \mathcal{V}]$, there exists a (unique in view of condition 1)!) word $u' \in W$ such that $u' \alpha_{\mathcal{V}} u$.

Let W be an arbitrary $\mathcal{V}$-cross section. For each permutation $\sigma \in \mathbf{Sym}(m)$, we define a transformation σ^* of the set W by letting $w\sigma^* \equiv \sigma(w)'$. Then it is easy to see that W becomes a $\mathbf{Sym}(m)$-set. Let $\mathcal{V}(k, m)$ denote the subvariety given within $\mathcal{V}$ by the identities $x_1 \ldots x_{k+1} = w = 0$, where w runs over the set of all words of length k depending on $\leq m$ letters.

THEOREM 10.1. *Let $\mathcal{V}$ be a homogeneous variety, m, k be positive integers, $m \leq k$, W be an arbitrary $\mathcal{V}$-cross section in the set $F[k, m; \mathcal{V}]$. Then the interval $[\mathcal{V}(k, m), \mathcal{V}(k, m-1)]$ is dually isomorphic to the lattice CP(W) of the coherent pairs of the $\mathbf{Sym}(m)$-set W.*

The approach to studying the lattices of nilsemigroup varieties developed in Chapter 1 is useful not only in the framework of the questions connected with the identities in varietal lattices. It essentially clarifies the structure of the lattices, helps to uniformly explain various known phenomena and opens the way to formulating (and solving) the problems which previously appeared to be unreachable. Thus, Theorem 10.1 should be also considered as on of the main achievements of the dissertations.

Chapter 2 is devoted to the proof of the necessity od the main theorem. In section 11 the necessity is established for its indicator version, in other words, we verify that all the varieties listed in the version have a non-modular subvariety lattice. Then we start analyzing the structure of the varieties containing none of these non-modular varieties. First we prove in section 12 that such a variety either satisfies one of the M1, M2, M2′, M3 of the main theorem or is equal to the join $\mathcal{M} \bigvee \mathcal{X}$, where $\mathcal{X}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and the variety $\mathcal{M}$ consists of nilsemigroups; then in section 13 we consider the case of nilsemigroup varieties. Thus, the necessity fort the structural version of the main theorem is proved also.

In sections 14–17 of Chapter 3 we consequently verify the modularity of the subvariety lattices of varieties satisfying each of the conditions M1–M4. This proves the sufficiency in both structural and indicator versions. In section 18 the equational version of the main theorem is proved and some corollaries are formulated which demonstrate the effectiveness of the description found. Here we exhibit only the most important

COROLLARY 18.2. *There exists an algorithm that, given any finite semigroup, decides whether the lattice of subvarieties of the variety is modular.*

We show also that the question of whether it is possible to algorithmically recognize the modularity of the subvariety lattice of a variety, given a finite basis for its identities, is equivalent to an old open problem of the theory of group varieties — namely to the problem of whether it is possible to algorithmically recognize abelian varieties among finitely based varieties of periodic groups.

Chapter 4 contains some of the most important applications of the main result and the methods developed for its proof. Section 19 is devoted to studying the permutability of fully invariant congruences on free semigroups. The connections between this property and identities in varietal lattices is well-known — e.g., it is the permutability of congruences on groups and rings that implies the modularity of the corresponding subvariety lattices. The permutability of "almost all" fully invariant congruences on free completely regular semigroups was established in the papers [5, 6] and applied there for another proof of the modularity of the lattice of completely regular semigroup varieties. Similar considerations (where permutability is substituted by its weaker version) are utilized in section 14 to prove the sufficiency of the condition M1. This makes our attention to the permutability of fully invariant congruences quite natural a priori; a posteriori it turns out that in the semigroup the property behaves in an unexpected way: it is connected rather with the distributivity than with the modularity.

We describe the varieties on whose free objects all fully invariant congruences commute:

THEOREM 19.1. *All fully invariant congruences on free semigroups of a variety $\mathcal{V}$ commute if and only if $\mathcal{V}$ satisfies one of the following conditions:*

P1 *$\mathcal{V}$ consists of completely simple semigroups;*

P2 *$\mathcal{V} = \mathcal{SL}$;*

P3 *$\mathcal{V}$ consists of nilsemigroups and satisfies one of the following identity systems (where in (37)–(42) ξ is one of the transpositions $\langle 12\rangle$ or $\langle 23\rangle$):*

$$x_1x_2x_3 = x_{1\xi}x_{2\xi}x_{3\xi},\ x^2y = yx^2,\ x^3y = 0; \tag{37}$$

$$x_1x_2x_3 = x_{1\xi}x_{2\xi}x_{3\xi},\ x^2y = yx^2,\ x^2y^2 = 0; \tag{38}$$

$$x_1x_2x_3 = x_{1\xi}x_{2\xi}x_{3\xi},\ x^2y = yx^2,\ x^3y = x^2y^2; \tag{39}$$

$$x_1x_2x_3 = x_{1\xi}x_{2\xi}x_{3\xi},\ x^2y = xy^2; \tag{40}$$

$$x_1x_2x_3 = x_{1\xi}x_{2\xi}x_{3\xi},\ xy^2 = 0; \tag{41}$$

$$x_1x_2x_3 = x_{1\xi}x_{2\xi}x_{3\xi},\ x^2y = 0; \tag{42}$$

$$x_1x_2x_3 = x_3x_2x_1,\ x^2y = xyx,\ x^3y = 0; \tag{43}$$

$$x_1x_2x_3 = x_3x_2x_1,\ x^2y = xyx,\ x^2y^2 = 0; \tag{44}$$

$$x_1x_2x_3 = x_3x_2x_1,\ x^2y = xyx,\ x^3y = x^2y^2; \tag{45}$$

$$x_1x_2x_3 = x_3x_2x_1,\ x^2y = yxy; \tag{46}$$

$$x_1x_2x_3 = x_3x_2x_1,\ xyx = 0; \tag{47}$$

$$x_1x_2x_3 = x_3x_2x_1,\ x^2y = 0; \tag{48}$$

$$x_1x_2x_3 = x_2x_3x_1,\ x^3y = 0; \tag{49}$$

$$x_1x_2x_3 = x_2x_3x_1,\ x^2y^2 = 0; \tag{50}$$

$$x_1x_2x_3 = x_2x_3x_1,\ x^3y = x^2y^2. \tag{51}$$

In section 20 semigroup varieties with a distributive subvariety lattice are investigated. The main result of section 20 gives a necessary condition of the distributivity which turns out to be also sufficient in the most cases.

THEOREM 20.1. *If the subvariety lattice of a semigroup variety $\mathcal{V}$ is distributive, then $\mathcal{V}$ satisfies one of the following conditions:*

D1 *the lattice of all group subvarieties of the variety $\mathcal{V}$ is distributive and, for every semigroup $S \in \mathcal{V}$, the semigroup $S^2 = \{st \mid s,t \in S\}$ is completely regular;*

D2 $\mathcal{V} = \mathcal{D} \bigvee \mathcal{X}$, *where the variety* $\mathcal{D}$ *consists of completely regular semigroups whose idempotents form a left self-distributive subsemigroup and* $\mathcal{X}$ *is one of the varieties* $\mathcal{P}$ *or* $\mathcal{Q}$*, and the lattice of all group subvarieties of the variety* $\mathcal{V}$ *is distributive;*

D2′ $\mathcal{V} = \mathcal{D} \bigvee \mathcal{X}$, *where the variety* $\mathcal{D}$ *consists of completely regular semigroups whose idempotents form a right self-distributive subsemigroup and* $\mathcal{X}$ *is one of the varieties* $\overleftarrow{\mathcal{P}}$ *or* $\overleftarrow{\mathcal{Q}}$*, and the lattice of all group subvarieties of the variety* $\mathcal{V}$ *is distributive;*

D3 $\mathcal{V} = \mathcal{N} \bigvee \mathcal{G} \bigvee \mathcal{X}$, *where the variety* $\mathcal{G}$ *consists of abelian groups,* $\mathcal{X}$ *is one of the varieties* $\mathcal{T}$, $\mathcal{SL}$ *or* $\mathcal{C}$*, and the variety* $\mathcal{N}$ *satisfies the identities* $x^2y = ztz = yx^2$ *and an identity of the kind* $x_1x_2x_3 = x_{1\xi}x_{2\xi}x_{3\xi}$, *where* ξ *is a non-trivial permutation;*

D4 $\mathcal{V} = \mathcal{M} \bigvee \mathcal{X}$, *where* $\mathcal{X}$ *is one of the varieties* $\mathcal{T}$ *or* $\mathcal{SL}$*, and the variety* $\mathcal{M}$ *consists of nilsemigroups and satisfies one of the identity systems (37)–(51).*

Conversely, if the variety $\mathcal{V}$ *satisfies either the condition D1 and idempotents form a subsemigroup in each semigroup from* $\mathcal{V}$ *or one of the conditions D2, D2′, D3 or D4, then the subvariety lattice of the variety* $\mathcal{V}$ *is distributive.*

Observe that here, by contrast with the description of modular varieties, some of the conditions are formulated “modulo groups”. Such the approach to the distributivity problem seems to be quite justified since there exist group varieties with a non-distributive subvariety lattice and the corresponding group problem still remains open (moreover, it appears to be rather far from a complete solution).

Comparison of the results of sections 19 and 20 leads to the aforementioned conclusion about a paradoxical character of the interconnection between the permutability of fully invariant congruences on free semigroups and identities in the lattices of semigroup varieties. We have, in particular,

COROLLARY 20.1. *The following are equivalent for a nilsemigroup variety* $\mathcal{V}$*:*

- $\mathcal{V}$ *has a distributive subvariety lattice;*
- *fully invariant congruences on the* $\mathcal{V}$*-free semigroup of countable rank commute.*

Section 21 is devoted to an investigation of another natural identity — the *arguesian law*. It is stronger than the modular identity in abstract lattices but it often turns out to be equivalent to the latter in congruence lattices (see, e.g., [2]). It is to expect therefore that the modular subvariety lattices of semigroup varieties will be arguesian. Actually it follows immediately from the proof of the main theorem that the varieties satisfying one of the conditions M2, M2′ or M3 have an arguesian subvariety lattice while the conditions M1 and M4 need a separate consideration. Section 21 contains a proof that the lattice of varieties consisting of semigroups with completely regular square (i.e. satisfying M1) is indeed arguesian.

Section 22 contains a brief survey of some further applications and consequences of the main results of the dissertation (proofs are not included there). The conclusion of the dissertation is devoted to discussing several interesting problems and open questions connected with the topic of the dissertation.

References

[1] Evans T., *The lattice of semigroup varieties*, Semigroup Forum, 2 (1971) 1–43.

[2] Freese R., Jónnson B., *Congruence modularity implies the Arguesian identity*, Algebra Universalis, 6 (1976) 225–228.

[3] Ivanov S.V., *On some questions of the theory of group varieties*, Internat. Conf. Algebra in Memoriam A.I.Mal'tsev, Abstracts of Reports on Group Theory, Novosibirsk, 1989, 49 [Russian].

[4] Pastijn F., *The lattice of completely regular semigroup varieties*, J. Austral. Math. Soc. Ser. A 49 (1990) 24–42.

[5] Pastijn F., *Commuting fully invariant congruences on free completely regular semigroups*, Trans. Amer. Math. Soc. 323 (1991) 79–92.

[6] Petrich M., Reilly N.R., *The modularity of the lattice of varieties of completely regular semigroups and related representations*, Glasgow Math. J. 32 (1990) 137–152.

[7] Polák L., *On varieties of completely regular semigroups* I,II,III, Semigroup Forum 32 (1985) 97–123; 36 (1987) 253–284; 37 (1988) 1–30.

[8] Schwabauer R. *The lattice of closed sets of commutative semigroup equations*, Ph.D. thesis. Univ. of Nebraska, Lincoln, NE, 1966.

[9] Shevrin L.N., Sukhanov, E.V., *Structural aspects of the theory of semigroup varieties*, Izv. Vyssh. Ucheb. Zaved. Matematika (1989) no.6, 3–39 [Russian; English translation Soviet Math. Izv. VUZ 33 (1990) no.6, 1–34].

[10] The Sverdlovsk notebook: unsolved problems in semigroup theory, 2nd ed., Ural State Univ., Sverdlovsk, 1979 [Russian].