YOUNG DIAGRAMS AND THE STRUCTURE OF THE LATTICE OF OVERCOMMUTATIVE SEMIGROUP VARIETIES

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Introduction

The lattice of semigroup varieties is a disjoint union of its ideal $\mathcal{P}$ consisting of all varieties of *periodic* semigroups and its coideal $\mathcal{Q}$ consisting of all varieties containing a non-periodic semigroup. Since every non-periodic semigroup contains a subsemigroup isomorphic to the additive semigroup of positive integers which generates the variety **Comm** of all commutative semigroups, $\mathcal{Q}$ is nothing but the principal coideal determined by **Comm** and varieties from $\mathcal{Q}$ are said to be *overcommutative*.

It is well known that the lattice of commutative semigroup varieties is rather bad from the point of view of the lattice theory — it contains an isomorphic copy of every finite lattice as follows from [1] and [6]. This implies that all non-trivial lattice conditions which are inherited by sublattices fail in this lattice and hence in the subvariety lattice of every overcommutative variety. Therefore the study of those conditions has always led to considering some sublattices of $\mathcal{P}$ only and almost nothing was known about the lattice structure and properties of $\mathcal{Q}$.

The aim of the present paper is to show that the lattice structure of $\mathcal{Q}$ is, however, surprisingly transparent. Namely, $\mathcal{Q}$ is a special subdirect product of a family of finite lattices; each of these finite lattices is dual to the congruence lattice of a G-set (that is a set on which a group G acts and which is considered as a unary algebra) where G runs over the set of all groups of permutations of rows of a Young diagram. These results admit several interesting applications connected, in particular, with the embeddability questions for the lattice $\mathcal{Q}$; in some other place we shall also demonstrate how they apply to study of the identities of this lattice (such as modularity or distributivity).

1 Derivation of balanced identities

Recall that an identity is said to be *balanced* if each letter occurs on both its sides the same number of times. It is well known that an identity is balanced if and only if it holds in the variety **Comm**. Thus, each identity satisfied by a overcommutative variety **V** is balanced.

Let u be a word. We denote by $\ell(u)$ the length of u. If $u \approx v$ is a balanced identity, then clearly $\ell(u) = \ell(v)$. This number is called *the length of the identity* $u \approx v$ and is denoted by $\ell(u \approx v)$. By $n(u)$ we denote the number of different letters occurring in u; if $u \approx v$ is a balanced identity, then clearly $n(u) = n(v)$ and we use the notation $n(u \approx v)$ for this number. The last parameter we associate with the word u is a partition of the number $\ell(u)$ into $n(u)$ parts. The partition (denoted by $\mathrm{part}(u)$) consists of the positive integers $\lambda_1, \lambda_2, \ldots, \lambda_{n(u)}$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n(u)}$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_{n(u)} = \ell(u)$; each λ_i is the number of occurrences of a letter in the word u. Again, if $u \approx v$ is a balanced identity, then $\mathrm{part}(u) = \mathrm{part}(v)$; so the notation $\mathrm{part}(u \approx v)$ is justified.

Let Σ be a system of identities. We shall use the following well-known description of the set of all consequences of Σ:

Proposition 1.1 *An identity $u \approx v$ follows from the system Σ if and only if there exists a sequence of words $w_0, w_1, \ldots, w_k$ such that $u \equiv w_0$, $v \equiv w_k$ and, for every $i = 0, 1, \ldots, k-1$, either $w_i \equiv w_{i+1}$ or there are words a_i, b_i, s_i, t_i and a substitution ζ_i such that $w_i \equiv a_i\zeta_i(s_i)b_i$, $w_{i+1} \equiv a_i\zeta_i(t_i)b_i$ and at least one of the identities $s_i \approx t_i$ and $t_i \approx s_i$ belongs to the system Σ.* ∎

We say that the sequence $w_0, w_1, \ldots, w_k$ is a *derivation* of $u \approx v$ from Σ and use the following notation:

$$u \equiv w_0 \xrightarrow{\Sigma} w_1 \xrightarrow{\Sigma} \ldots \xrightarrow{\Sigma} w_k \equiv v. \qquad (1)$$

If Σ is the set of all identities of a variety **X**, then we mark arrows by **X**'s instead of Σ's etc.

It follows immediately from Proposition 1.1 that if Σ consists of balanced identities, then each of its consequences is balanced as well. An easy but crucial for what follows corollary of the Proposition is that the parameters defined above are, in a sense, invariant with respect to the derivation process. More precisely, given a system Σ of balanced identities, we define

$$\ell(\Sigma) = \min\{\ell(s \approx t) |\ (s \approx t) \in \Sigma\},$$

$$n(\Sigma) = \max\{n(s \approx t) |\ (s \approx t) \in \Sigma,\ \ell(s \approx t) = \ell(\Sigma)\},$$

$$\mathrm{part}(\Sigma) = \{\lambda |\ \exists (s \approx t) \in \Sigma\ \ell(s \approx t) = \ell(\Sigma),\ n(s \approx t) = n(\Sigma),\ \mathrm{part}(s \approx t) = \lambda\}.$$

Then we have

Corollary 1.2 *Let Σ be a set of balanced identities and a non-trivial identity $u \approx v$ be a consequence of Σ. Then*

1. $\ell(u \approx v) \geq \ell(\Sigma)$;
2. $n(u \approx v) \leq n(\Sigma)$ *whenever* $\ell(u \approx v) = \ell(\Sigma)$;
3. $\mathrm{part}(u \approx v) \in \mathrm{part}(\Sigma)$ *whenever* $\ell(u \approx v) = \ell(\Sigma)$ *and* $n(u \approx v) = n(\Sigma)$.

Proof: Let (1) be a derivation of $u \approx v$ from Σ. All the identities $w_i \approx w_{i+1}$ in this derivation are balanced and therefore

$$\ell(u \approx v) = \ell(w_i \approx w_{i+1}),$$
$$n(u \approx v) = n(w_i \approx w_{i+1}),$$
$$\mathrm{part}(u \approx v) = \mathrm{part}(w_i \approx w_{i+1})$$

for all $i = 0, \ldots, k-1$. In other words, we can restrict ourselves to the case when the number k of steps of the derivation (1) is equal to 1. Thus, we may assume that there is a substitution ζ and words a, b, s, t such that $u \equiv a\zeta(s)b,\ v \equiv a\zeta(t)b$ and at least one of the identities $s \approx t$ and $t \approx s$ belongs to the system Σ. Then it is clear that

$$\ell(u \approx v) = \ell(u) = \ell(a\zeta(s)b) = \ell(a) + \ell(\zeta(s)) + \ell(b) \geq$$
$$\geq \ell(\zeta(s)) \geq \ell(s) = \ell(s \approx t) = \ell(t \approx s) \geq \ell(\Sigma). \qquad (2)$$

Now suppose that $\ell(u \approx v) = \ell(\Sigma)$. Then (2) implies that $\ell(a) = \ell(b) = 0$ (which means that the words a and b are empty) and $\ell(\zeta(s)) = \ell(s)$ which means that the substitution ζ maps every letter occurring in the word s to a letter. This implies that $n(\zeta(s)) \leq n(s)$ and therefore

$$n(u \approx v) = n(u) = n(\zeta(s)) \leq n(s) = n(s \approx t) = n(t \approx s) \leq n(\Sigma).$$

The latter inequality shows that $n(\zeta(s)) = n(s) = n(\Sigma)$ whenever $n(u \approx v) = n(\Sigma)$. This means that the restriction of ζ to the set of all letters occurring in s is a bijection of the set onto a set of $n(\Sigma)$ letters; in other words, it simply renames letters of s. Therefore $\mathrm{part}(\zeta(s)) = \mathrm{part}(s)$ and

$$\mathrm{part}(u \approx v) = \mathrm{part}(u) = \mathrm{part}(\zeta(s)) = \mathrm{part}(s) =$$
$$= \mathrm{part}(s \approx t) = \mathrm{part}(t \approx s) \in \mathrm{part}(\Sigma).$$

Thus, all statements of the corollary are verified. ∎

2 A subdirect decomposition of $\mathcal{Q}$

For each $n \geq 2$, let $\mathbf{C}_n$ denote the variety defined by all balanced identities of length $\geq n$. Clearly,

$$\mathbf{Comm} = \mathbf{C}_2 \subset \mathbf{C}_3 \subset \mathbf{C}_4 \subset \dots .$$

Now let $n \geq m \geq 1$. We denote by $\mathbf{C}_n^m$ the variety defined by all balanced identities of length $> n$ together with all balanced identities of length n depending on $\leq m$ letters. Clearly,

$$\mathbf{C}_n = \mathbf{C}_n^n \subset \mathbf{C}_n^{n-1} \subset \mathbf{C}_n^{n-2} \subset \dots \subset \mathbf{C}_n^2 \subset \mathbf{C}_n^1 = \mathbf{C}_{n+1}.$$

(The last equality follows from the fact that there are no non-trivial balanced identities on 1 letter.)

The aim of this Section is to show that the lattice $\mathcal{Q}$ is isomorphic to a subdirect product of its intervals of the kind $[\mathbf{C}_n^m, \mathbf{C}_n^{m-1}]$, where $2 \leq m \leq n$.

Recall that an element d of a lattice $(L; \vee, \wedge)$ is called *distributive* if, for all $x, y \in L$,

$$d \vee (x \wedge y) = (d \vee x) \wedge (d \vee y)$$

and *codistributive* if, for all $x, y \in L$,

$$d \wedge (x \vee y) = (d \wedge x) \vee (d \wedge y).$$

Lemma 2.1 *For each pair m, n such that $1 \leq m \leq n = 2, 3, \dots$, the variety $\mathbf{C}_n^m$ is both a distributive and codistributive element of the lattice $\mathcal{Q}$.*

Proof: Let $\mathbf{X}, \mathbf{Y} \in \mathcal{Q}$. To prove that

$$\mathbf{C}_n^m \vee (\mathbf{X} \wedge \mathbf{Y}) = (\mathbf{C}_n^m \vee \mathbf{X}) \wedge (\mathbf{C}_n^m \vee \mathbf{Y}), \tag{3}$$

it suffices to show that the left side of (3) contains the right one; this means that every identity $u \approx v$ holding in the variety $\mathbf{C}_n^m \vee (\mathbf{X} \wedge \mathbf{Y})$ holds also in the variety $(\mathbf{C}_n^m \vee \mathbf{X}) \wedge (\mathbf{C}_n^m \vee \mathbf{Y})$. Clearly, $\ell(u \approx v) \geq n$ and if $\ell(u \approx v) = n$, then $n(u \approx v) \leq m$. An arbitrary derivation of this identity from the identities of the varieties $\mathbf{X}$ and $\mathbf{Y}$ can be rewritten in the form:

$$u \equiv w_0 \xrightarrow{\mathbf{X}} w_1 \xrightarrow{\mathbf{Y}} \dots \longrightarrow w_k \equiv v, \tag{4}$$

where the identity $w_i \approx w_{i+1}$ holds in $\mathbf{X}$ for each even i and in $\mathbf{Y}$ for all odd i. All these identities are balanced and, for all i, we have $\ell(w_i) = \ell(u \approx v) \geq n$ and if $\ell(w_i) = \ell(u \approx v) = n$, then $n(w_i) = n(u \approx v) \leq m$. Therefore, for all i, they hold in the variety $\mathbf{C}_n^m$. Hence the identity $w_i \approx w_{i+1}$ holds in $\mathbf{C}_n^m \vee \mathbf{X}$ for each even

i and in $\mathbf{C}_n^m \vee \mathbf{Y}$ for all odd i and therefore we may consider (4) as a derivation of the identity $u \approx v$ from the identities of the variety $(\mathbf{C}_n^m \vee \mathbf{X}) \wedge (\mathbf{C}_n^m \vee \mathbf{Y})$. The equality (3) is proved.

To prove that

$$\mathbf{C}_n^m \wedge (\mathbf{X} \vee \mathbf{Y}) = (\mathbf{C}_n^m \wedge \mathbf{X}) \vee (\mathbf{C}_n^m \wedge \mathbf{Y}), \tag{5}$$

it is enough to check that right side of (5) contains the left one. In other words, we have to show that every identity $u \approx v$ holding in the variety $(\mathbf{C}_n^m \wedge \mathbf{X}) \vee (\mathbf{C}_n^m \wedge \mathbf{Y})$ holds also in the variety $\mathbf{C}_n^m \wedge (\mathbf{X} \vee \mathbf{Y})$. If either $\ell(u \approx v) > n$ or $\ell(u \approx v) = n$ and $n(u \approx v) \leq m$, then $u \approx v$ holds even in $\mathbf{C}_n^m$ and everything is clear. Otherwise, deriving $u \approx v$ from the identities of the variety $\mathbf{C}_n^m \wedge \mathbf{X}$, we may apply no identity of $\mathbf{C}_n^m$ since all these identities are either "too long" or have too few letters, as Corollary 1.2 shows. Thus, only identities of $\mathbf{X}$ may be used in such a derivation and this means that $u \approx v$ holds in the variety $\mathbf{X}$. Analogously, $u \approx v$ holds in the variety $\mathbf{Y}$. Therefore it holds in the join $\mathbf{X} \vee \mathbf{Y}$ and hence in the variety $\mathbf{C}_n^m \wedge (\mathbf{X} \vee \mathbf{Y})$ too. We have got the equality (5). ■

Proposition 2.2 *The lattice $\mathcal{Q}$ is isomorphic to a subdirect product of its intervals of the kind $[\mathbf{C}_n^m, \mathbf{C}_n^{m-1}]$, where $2 \leq m \leq n$.*

Proof: Define a mapping $\varphi_n^m : \mathcal{Q} \longrightarrow [\mathbf{C}_n^m, \mathbf{C}_n^{m-1}]$ by the rule:

$$\varphi_n^m(\mathbf{X}) = (\mathbf{X} \vee \mathbf{C}_n^m) \wedge \mathbf{C}_n^{m-1}.$$

Lemma 2.1 immediately implies that φ_n^m is a lattice homomorphism for each pair m, n such that $2 \leq m \leq n$. Since the restriction of φ_n^m to $[\mathbf{C}_n^m, \mathbf{C}_n^{m-1}]$ is the identical mapping, this homomorphism is surjective. It remains to verify that the homomorphisms φ_n^m separate $\mathcal{Q}$; that is, for every $\mathbf{X}, \mathbf{Y} \in \mathcal{Q}$ such that $\mathbf{X} \neq \mathbf{Y}$, there exist m, n such that $\varphi_n^m(\mathbf{X}) \neq \varphi_n^m(\mathbf{Y})$. Indeed, without loss of generality we may assume that there is an identity $u \approx v$, which holds in $\mathbf{X}$ and fails in $\mathbf{Y}$. Let $n = \ell(u \approx v)$, $m = n(u \approx v)$. Then the identity $u \approx v$ holds in the variety $\mathbf{C}_n^m$ and hence it is also true in the variety $\varphi_n^m(\mathbf{X}) = (\mathbf{X} \vee \mathbf{C}_n^m) \wedge \mathbf{C}_n^{m-1}$. Suppose that this identity holds in the variety $\varphi_n^m(\mathbf{Y}) = (\mathbf{Y} \vee \mathbf{C}_n^m) \wedge \mathbf{C}_n^{m-1}$. Corollary 1.2 shows that no identity of the variety $\mathbf{C}_n^{m-1}$ may take part in the derivation of $u \approx v$ from the identities of $(\mathbf{Y} \vee \mathbf{C}_n^m) \wedge \mathbf{C}_n^{m-1}$ which means that $u \approx v$ should be satisfied by $\mathbf{Y} \vee \mathbf{C}_n^m$ and hence by $\mathbf{Y}$, a contradiction. Thus, $u \approx v$ fails in the variety $\varphi_n^m(\mathbf{Y})$ and $\varphi_n^m(\mathbf{X}) \neq \varphi_n^m(\mathbf{Y})$. ■

It is easy to see that all the intervals $[\mathbf{C}_n^m, \mathbf{C}_n^{m-1}]$ are finite. Indeed, every variety $\mathbf{X} \in [\mathbf{C}_n^m, \mathbf{C}_n^{m-1}]$ is defined within the variety $\mathbf{C}_n^{m-1}$ by some system Σ of identities of the length n depending on m letters. Renaming the letters, we may assume that each identity in Σ depends on the letters $x_1, x_2, \ldots, x_m$. Since there are only finitely many words of length n over an m-element alphabet, there are only finitely many possibilities for Σ and hence for $\mathbf{X}$. This remark and Proposition 2.2 imply

Corollary 2.3 *The lattice $\mathcal{Q}$ is residually finite.* ■

Corollary 2.4 *The dual of the lattice EqM of all equivalences on an infinite set M cannot be embedded into the lattice $\mathcal{Q}$.*

Proof: A sublattice of a residually finite lattice is residually finite. Since the lattice EqM is simple [2, Theorem IV.4.2], it fails to be residually finite whenever M is infinite. The result now follows from Corollary 2.3. ■

Corollary 2.4 strengthens a result by McNulty [4] who proved that, for no infinite M, the dual of EqM can be embedded into $\mathcal{Q}$ *as an interval*. We note that $\mathcal{Q}$ contains an isomorphic copy of the dual of the lattice EqM for any finite set M, see Corollary 4.5 below.

3 A direct decomposition of $[\mathbf{C}_n^m, \mathbf{C}_n^{m-1}]$

We fix now integers m and n such that $2 \leq m \leq n$. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of n into m parts. We denote by $\mathbf{C}_n^m(\lambda)$ the variety given within the variety $\mathbf{C}_n^{m-1}$ by all balanced identities $u \approx v$ such that

$$\ell(u \approx v) = n,\ n(u \approx v) = m,\ \mathrm{part}(u \approx v) = \lambda.$$

Our next step in describing the lattice of all overcommutative semigroup varieties is the following

Proposition 3.1 *The interval $[\mathbf{C}_n^m, \mathbf{C}_n^{m-1}]$ is isomorphic to a direct product of the intervals $[\mathbf{C}_n^m(\lambda), \mathbf{C}_n^{m-1}]$ where λ runs over the set of all partitions of n into m parts.*

Proof: Let

$$L = \prod_{\lambda} [\mathbf{C}_n^m(\lambda), \mathbf{C}_n^{m-1}].$$

Consider two mappings $\alpha : [\mathbf{C}_n^m, \mathbf{C}_n^{m-1}] \longrightarrow L$ and $\beta : L \longrightarrow [\mathbf{C}_n^m, \mathbf{C}_n^{m-1}]$ defined by the rules

$$\alpha(\mathbf{X}) = (\ldots, \mathbf{X} \vee \mathbf{C}_n^m(\lambda), \ldots);$$

$$\beta(\ldots, \mathbf{Y}_\lambda, \ldots) = \bigwedge_{\lambda} \mathbf{Y}_\lambda.$$

We are going to prove that α and β are mutually inverse bijections. Since both α and β are obviously order-preserving, this will imply that $[\mathbf{C}_n^m, \mathbf{C}_n^{m-1}]$ and L

are isomorphic as posets and the latter is well known to be equivalent to their isomorphism as lattices. To prove that $\beta(\alpha(\mathbf{X})) = \mathbf{X}$, we have to verify that

$$\bigwedge_{\lambda} (\mathbf{X} \vee \mathbf{C}_n^m(\lambda)) = \mathbf{X}. \tag{6}$$

It suffices to show that the left side of (6) is contained in its right side. Let us take an arbitrary identity $u \approx v$ holding in $\mathbf{X}$; we have to check that it holds in the left side of (6). If $\ell(u \approx v) > n$ or $\ell(u \approx v) = n,\ n(u \approx v) < m$, then $u \approx v$ is true even in the variety $\mathbf{C}_n^{m-1}$. Thus, we may assume that $\ell(u \approx v) = n,\ n(u \approx v) = m$ and then $u \approx v$ will hold in the variety $\mathbf{C}_n^m(\lambda)$ where $\lambda = \mathrm{part}(u \approx v)$. Hence $u \approx v$ is also true in $\mathbf{X} \vee \mathbf{C}_n^m(\lambda)$ and in $\bigwedge_\lambda \mathbf{X} \vee \mathbf{C}_n^m(\lambda))$.

It remains to check that $\alpha(\beta(\ldots, \mathbf{Y}_\lambda, \ldots)) = (\ldots, \mathbf{Y}_\lambda, \ldots)$, that is,

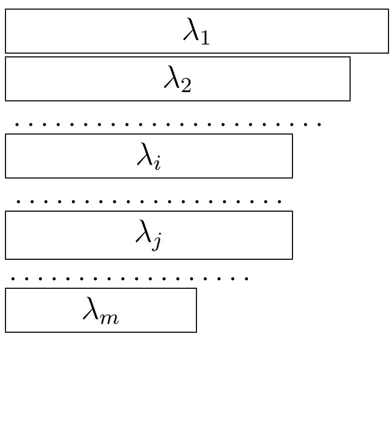
$$\mathbf{C}_n^m(\mu) \vee \bigwedge_{\lambda} \mathbf{Y}_\lambda = \mathbf{Y}_\mu. \tag{7}$$

Here it is sufficient to prove that the left side of (7) contains the right one. Let $u \approx v$ be an identity holding in $\mathbf{C}_n^m(\mu) \vee \bigwedge_\lambda \mathbf{Y}_\lambda$; we want to show that it holds in the variety $\mathbf{Y}_\mu$. This is clear if $\ell(u \approx v) > n$ or $\ell(u \approx v) = n,\ n(u \approx v) < m$ since the variety $\mathbf{C}_n^{m-1}$ satisfies $u \approx v$ in that case. If $\ell(u \approx v) = n$ and $n(u \approx v) = m$, then $\mathrm{part}(u \approx v) = \mu$ for $u \approx v$ should be true in the variety $\mathbf{C}_n^{m-1}(\mu)$. Corollary 1.2 implies that deriving $u \approx v$ from the identities of the varieties $\mathbf{Y}_\lambda$, we may apply no identity of a variety $\mathbf{Y}_\lambda$ where $\lambda \neq \mu$. Therefore $u \approx v$ holds in $\mathbf{Y}_\mu$ and we have got the equality (7). ■

4 Isomorphism theorem

Proposition 2.2 and 3.1 reduce the problem of describing the lattice $\mathcal{Q}$ to the problem of determining the structure of its intervals of the kind $[\mathbf{C}_n^m(\lambda), \mathbf{C}_n^{m-1}]$. Let us fix a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ of n. We associate with λ the following subgroup G_λ of the group $\mathbf{Sym}_m$ of all permutation of the set $\{1, \ldots, m\}$:

$$G_\lambda = \{\pi \mid i\pi = j \Rightarrow \lambda_i = \lambda_j\}.$$



If we, as it is usual, visualize λ by means of a Young diagram, then we may identify G_λ with the group of all permutations on the set of rows of the diagram which preserve the diagram (or, more accurately, the form of the diagram). The group structure of G_λ is rather transparent: it is isomorphic to a direct product of the symmetric groups $\mathbf{Sym}_{\kappa_k}$, where κ_k is the number of rows of length k in the diagram.

Let $W[n, m, \lambda]$ denote the set of all words w of length n such that the letter x_i occurs in w exactly λ_i times for all $i = 1, \dots, m$. (So, in particular, $\mathrm{part}(w) = \lambda$ for every $w \in W[n, m, \lambda]$.) There is a natural action of G_λ on $W[n, m, \lambda]$ by permuting indices of the letters[1]:

$$\pi(w(x_1, \dots, x_m)) = w(x_{1\pi}, \dots, x_{m\pi}).$$

This action is obviously a representation of G_λ by transformations of the set $W[n, m, \lambda]$. Following [3], we consider triples of the kind (M, G, φ), where G is a group and $\varphi : G \longrightarrow \mathbf{Sym}(M)$ is a representation of G by transformations of a set M, as unary algebras with the carrier M and the set of unary operations G and call these unary algebras *G-sets*. Thus, $W[n, m, \lambda]$ is a G_λ-set. Below, speaking about congruences on $W[n, m, \lambda]$, we ever have in mind this unary structure.

Theorem 4.1 *For any pair m, n such that $2 \le m \le n$ and for any partition λ of n into m parts, the interval $[\mathbf{C}_n^m(\lambda), \mathbf{C}_n^{m-1}]$ is antiisomorphic to the congruence lattice of the G_λ-set $W[n, m, \lambda]$.*

Proof: We will make use of the well known antiisomorphism between the lattice of semigroup varieties and the lattice of fully invariant congruences on the free semigroup F over the alphabet $\{x_1, x_2, \dots\}$. If ρ_n^{m-1} (resp., $\rho_n^m(\lambda)$) is the fully invariant congruence corresponding to the variety $\mathbf{C}_n^{m-1}$ (resp., $\mathbf{C}_n^m(\lambda)$), then the interval $[\mathbf{C}_n^m(\lambda), \mathbf{C}_n^{m-1}]$ is antiisomorphic to the interval $[\rho_n^{m-1}, \rho_n^m(\lambda)]$ of the latter lattice. Thus it suffices to prove that the interval $[\rho_n^{m-1}, \rho_n^m(\lambda)]$ is isomorphic to the lattice $\mathrm{Con}W[n, m, \lambda]$.

For a fully invariant congruence $\rho \in [\rho_n^{m-1}, \rho_n^m(\lambda)]$, let $\rho{\downarrow}$ denote the restriction of ρ to $W[n, m, \lambda]$. Since permutations from G_λ obviously expand to automorphisms of F, the relation $\rho{\downarrow}$ is a congruence on $W[n, m, \lambda]$. Conversely, for a congruence γ on $W[n, m, \lambda]$, let $\gamma{\uparrow}$ be the join (in the lattice of fully invariant congruences on F) of the fully invariant congruence generated by γ with the congruence ρ_n^{m-1}. Since both the mappings $\rho \mapsto \rho{\downarrow}$ and $\gamma \mapsto \gamma{\uparrow}$ are obviously order-preserving, it suffices to verify that the are mutually inverse bijection. This means that, for each $\rho \in [\rho_n^{m-1}, \rho_n^m(\lambda)]$, $\rho{\downarrow}{\uparrow} = \rho$ and, for each $\gamma \in \mathrm{Con}W[n, m, \lambda]$, $\gamma{\uparrow}{\downarrow} = \gamma$.

Since ρ is a fully invariant congruence containing both $\rho{\downarrow}$ and ρ_n^{m-1}, it contains the fully invariant congruence $\rho{\downarrow}{\uparrow}$ they generate. Conversely, suppose that $(u, v) \in \rho \backslash \rho_n^{m-1} \subseteq \rho_n^m(\lambda) \backslash \rho_n^{m-1}$. Then u and v are words of the length n depending on the same m letters (say, $y_1, \dots, y_m$) and $\mathrm{part}(u) = \mathrm{part}(v) = \lambda$. Let ζ be an automorphism of the semigroup F such that $\zeta(y_i) = x_i$ for all $i = 1, \dots, m$. Applying

[1] We denote by the same letter the permutation itself and the transformation of $W[n, m, \lambda]$ it induces, but we write permutations on the right side from the argument while the transformations are written on the left side.

it to u and v, we get words $\zeta(u)$ and $\zeta(v)$ which lie in the set $W[n, m, \lambda]$ and are ρ-related. Thus, $(\zeta(u), \zeta(v)) \in \rho\!\downarrow$ which means that $(u, v) = (\zeta^{-1}(\zeta(u)), \zeta^{-1}(\zeta(v)))$ belongs to the fully invariant congruence generated by $rho\!\downarrow$.

It is clear that $\gamma\!\uparrow\downarrow$ contains γ. Conversely, let $(u, v) \in \gamma\!\uparrow\downarrow$. This means that the words u and v lie in $W[n, m, \lambda]$ and $(u, v) \in \gamma\!\uparrow$. If we consider γ as a set of identities, then $\gamma\!\uparrow$ is nothing but the set of all consequences of γ and therefore its structure is described by Proposition 1.1. This means that there exists a derivation

$$u \equiv w_0 \xrightarrow{\gamma} w_1 \xrightarrow{\gamma} \dots \xrightarrow{\gamma} w_k \equiv v, \tag{8}$$

of $u \approx v$ from γ. All the identities $w_i \approx w_{i+1}$ in (8) are balanced and therefore $\ell(w_i) = \ell(u) = n$, $n(w_i) = n(u) = m$, and $\text{part}(w_i) = \text{part}(u) = \lambda$ for all $i = 0, \dots, k$. This means that $w_i \in W[n, m, \lambda]$ for all $i = 0, \dots, k$; in other words, it suffices to consider only the case when the number k of steps of the derivation (8) is equal to 1. Then there are words a, b, s, t and a substitution ζ such that $u \equiv a\zeta(s)b$, $v \equiv a\zeta(t)b$ and $s\gamma t$. Since $\ell(u) = \ell(s)$, the words a and b should be empty and ζ should map each of the letters $x_1, \dots, x_m$ on a letter; furthermore, since u depends on the same letters $x_1, \dots, x_m$, the restriction of ζ to the set $\{x_1, \dots, x_m\}$ is a permutation of this set. Therefore ζ restricted to the set $W[n, m, \lambda]$ belongs to the group G_λ. Now we can finally utilize the fact that γ is a congruence of $W[n, m, \lambda]$ as a G_λ-set. This means, in particular, that $s\gamma t$ implies $u \equiv \zeta(s)\gamma\zeta(t) \equiv v$. ■

Consider a very special but still interesting case when $n = m$. In this case there exists only one partition λ of n into m parts (namely, $n = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}}$). Clearly, the set $W[n, m, \lambda]$ is merely the set M_n of all words of the kind $x_{1\pi} \cdots x_{m\pi}$, where π runs over the symmetric group $\mathbf{Sym}_m$ of all permutations of the set $1, \dots, m$, $\mathbf{C}_n^m(\lambda) = \mathbf{C}_m^m$, and the group G_λ coincides with $\mathbf{Sym}_m$. Since the group $\mathbf{Sym}_m$ acts on the $\mathbf{Sym}_m$-set M_n transitively, we may apply the following well known result to calculate the congruence lattice of this unary algebra:

Lemma 4.2 (see, for instance, [3, Lemma 4.20]) *If A is a transitive G-set, then, for any $a \in A$, the lattice* Con$A$ *is isomorphic to the interval $[Stab_G(a), G]$ in the subgroup lattice of G, where $Stab_G(a) = \{g \in G \mid g(a) = a\}$.* ■

Combining Theorem 4.1 and Lemma 4.2, we immediately obtain

Corollary 4.3 *For any $m \geq 2$, the interval $[\mathbf{C}_m^m, \mathbf{C}_m^{m-1}]$ is antiisomorphic to the subgroup lattice of the symmetric group $\mathbf{Sym}_m$.* ■

Since every finite lattice can be embedded into the subgroup lattice of a finite group (see [6]) and every finite group can be identified with a subgroup of $\mathbf{Sym}_m$, we get also

Corollary 4.4 *Every finite lattice can be embedded into the lattice $\mathcal{Q}$ of all overcommutative semigroup varieties.* ■

We note that not every countable lattice can be embedded into $\mathcal{Q}$ as it follows from Corollary 2.3. Our results also imply that the question whether every finite lattice can be embedded into the lattice $\mathcal{Q}$ *as an interval* is connected with the well known open problem in universal algebra of whether every finite lattice is isomorphic to the congruence lattice of a finite algebra. (Indeed, in view of a result by Pálfy and Pudlák [5], the latter property is equivalent to the property that every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.) We have, however, the following partial result related to interval embeddings into $\mathcal{Q}$:

Corollary 4.5 *For any finite set M the lattice $\mathcal{Q}$ contains an interval which is antiisomorphic to the lattice EqM of all equivalences on M.*

Proof: Let $n = m(m+1)/2$ and denote by λ the partition $n = 1 + 2 + \cdots + m$. Since the numbers λ_i are pairwise different, the group G_λ is trivial and therefore every equivalence on the set $W[n, m, \lambda]$ is a congruence of this G_λ-set. By Theorem 4.1 the interval $[\mathbf{C}_n^m(\lambda), \mathbf{C}_n^{m-1}]$ is antiisomorphic to the lattice $EqW[n, m, \lambda]$. Since the cardinality of $W[n, m, \lambda]$ increases when $m \to \infty$, we can identify any finite set M with a subset of $W[n, m, \lambda]$ for some m and it is clear that $EqW[n, m, \lambda]$ contains an interval isomorphic to EqM for each $M \subseteq W[n, m, \lambda]$ ■

5 The lattice of overcommutative subvarieties of a variety

Let $\mathbf{V}$ be a variety containing the variety **Comm** of all commutative semigroups. We call the interval [**Comm**,**V**] of the lattice of all semigroup varieties *the lattice of overcommutative subvarieties* of $\mathbf{V}$ and denote it by $L_o(\mathbf{V})$. The aim of this Section is to show that most of the previous results can be "relativized" to provide us with a description of the lattice $L_o(\mathbf{V})$ for an arbitrary overcommutative variety $\mathbf{V}$. This relativization can be carried out in two (of course, equivalent) ways. First, we can work within the free semigroup $F(\mathbf{V})$ of the variety $\mathbf{V}$ instead of the free semigroup F. All we need is to define the parameters $\ell(s)$, $n(s)$, and part(s) for elements of $F(\mathbf{V})$ and this can be done straightforwardly by letting $\ell(s) = \ell(u)$, $n(s) = n(u)$, and $\text{part}(s) = \text{part}(u)$, where $u \in F$ is an arbitrary preimage of $s \in F(\mathbf{V})$. (Such the definition is correct because no application of a balanced identity can change the value of the parameters in question.) After $\ell(s)$, $n(s)$, and part(s) have been defined, all the constructions and principal results of Sections 2–4 transfer to the lattice $L_o(\mathbf{V})$ automatically for we can repeat all the arguments literally. The point of view is quite natural but we shall go another way. Namely, we prefer to keep working within F rather than within $F(\mathbf{V})$. The use of "proper" words makes the constructions more explicit, which is important if one wants to calculate the lattice $L_o(\mathbf{V})$ practically. The disadvantage of this approach consists in the necessity of reproving the "relative" variants of our previous results. These proofs, however, are not too hard.

Denote by $\mathbf{V}_n^m$ the intersection $\mathbf{V}\wedge\mathbf{C}_m^n$.

Proposition 5.1 *The lattice $L_o(\mathbf{V})$ is isomorphic to a subdirect product of their intervals of the kind $[\mathbf{V}_n^m,\mathbf{V}_n^{m-1}]$, where $m,n=2,3,\dots$ and $m\leq n$.*

Proof: Restricting to the lattice $L_o(\mathbf{V})$ the homomorphisms

$$\varphi_n^m : \mathbf{X} \longmapsto (\mathbf{X}\vee\mathbf{C}_n^m)\wedge\mathbf{C}_n^{m-1}$$

constructed in the proof of Proposition 2.2, we obtain a decomposition of this lattice into a subdirect product of the intervals of the kind $[\varphi_n^m(\mathbf{Comm}),\varphi_n^m(\mathbf{V})]$. We note that

$$\varphi_n^m(\mathbf{Comm}) = (\mathbf{Comm}\vee\mathbf{C}_n^m)\wedge\mathbf{C}_n^{m-1} = \mathbf{C}_n^m,$$

$$\begin{aligned}\varphi_n^m(\mathbf{V}) = (\mathbf{V}\vee\mathbf{C}_n^m)\wedge\mathbf{C}_n^{m-1} = &\text{ in view of Lemma 2.1}\\ &= (\mathbf{V}\wedge\mathbf{C}_n^{m-1})\vee\mathbf{C}_n^m = \mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m.\end{aligned}$$

Thus, in order to prove the Proposition, it remains to check that the intervals $[\mathbf{C}_n^m,\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m]$ and $[\mathbf{V}_n^m,\mathbf{V}_n^{m-1}]$ are isomorphic.

Using Lemma 2.1 we get that the mapping $\mathbf{X}\mapsto\mathbf{X}\vee\mathbf{C}_n^m$ is a homomorphism of the latter interval onto the former one. To show that this homomorphism is 1–1, take two varieties $\mathbf{X},\mathbf{Y} \in [\mathbf{V}_n^m,\mathbf{V}_n^{m-1}]$. They are defined within the variety $\mathbf{V}_n^{m-1}$ by identities of length n depending on m letters; therefore, if $\mathbf{X}\neq\mathbf{Y}$, then without loss of generality we may assume that there is an identity $u\approx v$ such that $\ell(u\approx v)=m,\ n(u\approx v)=m$, which holds in $\mathbf{X}$ and fails in $\mathbf{Y}$. Since $u\approx v$ holds in the variety $\mathbf{C}_n^m$, it is also true in the variety $\mathbf{X}\vee\mathbf{C}_n^m$ and would we suppose that $\mathbf{X}\vee\mathbf{C}_n^m=\mathbf{Y}\vee\mathbf{C}_n^m$, we would immediately obtain a contradiction. ■

Let λ be a partition of n into m parts. We denote by $\mathbf{V}_n^m(\lambda)$ the intersection $\mathbf{V}\wedge\mathbf{C}_m^n(\lambda)$.

Proposition 5.2 *The interval $[\mathbf{V}_n^m,\mathbf{V}_n^{m-1}]$ is isomorphic to a direct product of the intervals $[\mathbf{V}_n^m(\lambda),\mathbf{V}_n^{m-1}]$, where λ runs over the set of all partition of n into m parts.*

Proof: Here we make use of the isomorphism $[\mathbf{V}_n^m,\mathbf{V}_n^{m-1}] \cong \mathbf{C}_n^m,\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m]$ constructed in the proof of Proposition 5.1. In the proof of Proposition 3.1 we showed that the mapping

$$\alpha : [\mathbf{C}_n^m,\mathbf{C}_n^{m-1}] \longrightarrow \prod_\lambda [\mathbf{C}_n^m(\lambda),\mathbf{C}_n^{m-1}]$$

defined by $\alpha(\mathbf{X}) = (\ldots,\mathbf{X}\vee\mathbf{C}_n^m(\lambda),\ldots)$ is an isomorphism. Restricting it to the interval $[\mathbf{C}_n^m,\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m]$, we obtain an isomorphism of this interval *into* the direct product of the intervals $[\mathbf{C}_n^m(\lambda),\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m(\lambda)]$, where λ runs over the set of all partition of n into m parts. Repeating almost literally the proof of the isomorphism $[\mathbf{V}_n^m,\mathbf{V}_n^{m-1}] \cong [\mathbf{C}_n^m,\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m]$, one gets that, for each λ, there is also an isomorphism between the intervals $[\mathbf{C}_n^m(\lambda),\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m(\lambda)]$ and $[\mathbf{V}_n^m(\lambda),\mathbf{V}_n^{m-1}]$. Therefore, to prove our Proposition, it remains to verify that in fact α restricted to $[\mathbf{C}_n^m,\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m]$ maps this interval *onto* $\prod_\lambda [\mathbf{C}_n^m(\lambda),\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m(\lambda)]$.

Indeed, let $(\ldots,\mathbf{Y}_\lambda,\ldots) \in \prod_\lambda [\mathbf{C}_n^m(\lambda),\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m(\lambda)]$. Put $\mathbf{Y}= \bigwedge_\lambda \mathbf{Y}_\lambda$. We showed in the proof of Proposition 3.1 that $\alpha(\mathbf{Y}) = (\ldots,\mathbf{Y}_\lambda,\ldots)$. But

$$\mathbf{Y}= \bigwedge_\lambda \mathbf{Y}_\lambda \subseteq \bigwedge_\lambda \mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m(\lambda)) = \alpha^{-1}(\ldots,\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m(\lambda),\ldots) = \\ = \alpha^{-1}(\alpha(\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m)) = \mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m,$$

whence $\mathbf{Y}\in [\mathbf{C}_n^m,\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m]$. Thus, we proved that the preimage under α of an arbitrary element of $\prod_\lambda [\mathbf{C}_n^m(\lambda),\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m(\lambda)]$ lies in $[\mathbf{C}_n^m,\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m]$. ■

It remains to "relativize" Theorem 4.1. Let ρ be the fully invariant congruence on F corresponding to the variety $\mathbf{V}$. Given a partition λ of n into m parts, we define the set $W[n,m,\lambda;\mathbf{V}]$ as an arbitrary set of representatives of $\rho|_{W[n,m,\lambda]}$-classes, where $\rho|_{W[n,m,\lambda]}$ is ρ restricted to the set $W[n,m,\lambda]$. This means that $W[n,m,\lambda;\mathbf{V}]$ enjoys two properties:

1. if $u,v \in W[n,m,\lambda;\mathbf{V}]$ and $u \neq v$, then u and v lie in different ρ-classes;
2. for every $w \in W[n,m,\lambda]$, there exists a (unique in view of the previous property) $\bar{w} \in W[n,m,\lambda;\mathbf{V}]$ such that $w\rho\bar{w}$.

For every $\pi \in G_\lambda$ we define a transformation $\hat{\pi}$ of the set $W[n,m,\lambda;\mathbf{V}]$ by letting $\hat{\pi}(w) = \overline{\pi(w)}$ for all $w \in W[n,m,\lambda;\mathbf{V}]$.

Lemma 5.3 *The mapping $\pi \mapsto \hat{\pi}$ is a representation of the group G_λ by transformations of the set $W[n,m,\lambda;\mathbf{V}]$.*

Proof: We have to check that $\widehat{\pi\sigma} = \hat{\pi}\hat{\sigma}$ for any $\pi,\sigma \in G_\lambda$. Indeed, for an arbitrary word $w \in W[n,m,\lambda;\mathbf{V}]$, we have $\hat{\pi}(w)\rho\pi(w)$ by the definition. Applying σ and taking into account that ρ is fully invariant, we obtain

$$(\pi\sigma)(w) \equiv \sigma(\pi(w))\rho\sigma(\hat{\pi}(w))\rho\hat{\sigma}(\hat{\pi}(w)).$$

On the other hand, $(\pi\sigma)(w)\rho\widehat{\pi\sigma}(w)$. This yields $\hat{\sigma}(\hat{\pi}(w))\rho\widehat{\pi\sigma}(w)$. Since both the words $\hat{\sigma}(\hat{\pi}(w))$ and $\widehat{\pi\sigma}(w)$ belong to the set $W[n,m,\lambda;\mathbf{V}]$, property 1 of this set implies that $\hat{\sigma}(\hat{\pi}(w)) \equiv \widehat{\pi\sigma}(w)$ and therefore $\hat{\pi}\hat{\sigma} = \widehat{\pi\sigma}$. ∎

Lemma 5.3 shows that we can consider $W[n,m,\lambda;\mathbf{V}]$ as a G_λ-set. Clearly, it is isomorphic (as a G_λ-set) to the quotient of $W[n,m,\lambda]$ over the restriction $\rho\!\downarrow$ of ρ to $W[n,m,\lambda]$. This remark will be useful in proving

Theorem 5.4 *For any pair m,n such that $2 \le m \le n$ and for any partition λ of n into m parts, the interval $[\mathbf{V}_n^m(\lambda),\mathbf{V}_n^{m-1}]$ is antiisomorphic to the congruence lattice of the G_λ-set $W[n,m,\lambda;\mathbf{V}]$.*

Proof: We have mentioned in the proof of Proposition 5.2 that the intervals $[\mathbf{V}_n^m(\lambda),\mathbf{V}_n^{m-1}]$ and $[\mathbf{C}_n^m,\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m]$ are isomorphic. The latter interval is a principal ideal of the interval $[\mathbf{C}_n^m,\mathbf{C}_n^{m-1}]$, which is antiisomorphic to the lattice $\mathrm{Con}W[n,m,\lambda]$ according to Theorem 4.1. Hence $[\mathbf{C}_n^m,\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m]$ is antiisomorphic to the principal coideal of $\mathrm{Con}W[n,m,\lambda]$ determined by the congruence γ on the set $W[n,m,\lambda]$ corresponding to the variety $\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m$. From the proof of Theorem 4.1 it is clear that γ is nothing but the restriction of the fully invariant congruence on F corresponding to $\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m$ to $W[n,m,\lambda]$. It is easy to check that the latter congruence and the fully invariant congruence ρ corresponding to $\mathbf{V}$ have the same restrictions to the set $W[n,m,\lambda]$ whence we can identify γ with the congruence $\rho\!\downarrow$. We have noticed (just before the formulation of this Theorem) that there is an isomorphism of G_λ-sets $W[n,m,\lambda;\mathbf{V}]$ and $W[n,m,\lambda]/\rho\!\downarrow$. By a standard result of universal algebra (see, for instance, [3, Theorem 4.12]) this implies that the principal coideal $[\rho\!\downarrow)$ of $\mathrm{Con}W[n,m,\lambda]$ is isomorphic to the lattice $\mathrm{Con}W[n,m,\lambda;\mathbf{V}]$. Thus, we have found the following chain of mappings

$$[\mathbf{V}_n^m(\lambda),\mathbf{V}_n^{m-1}] \xrightarrow{\xi} [\mathbf{C}_n^m,\mathbf{V}_n^{m-1}\vee\mathbf{C}_n^m] \xrightarrow{\chi} [\rho\downarrow) \xrightarrow{\psi} \mathrm{Con}W[n,m,\lambda;\mathbf{V}]$$

in which ξ and ψ are isomorphisms and χ is an antiisomorphism. Composing ξ, χ, and ψ we get the desired antiisomorphism of $[\mathbf{V}_n^m(\lambda),\mathbf{V}_n^{m-1}]$ onto $\mathrm{Con}W[n,m,\lambda;\mathbf{V}]$. ∎

6 Final remarks

In conclusion we note that, as the reader might already observe, we never used the associativity above. Thus our principal results remain true for the varieties of *groupoids* containing the variety **Comm** and — mutatis mutandi — for any class of universal algebras where the notion of a balanced identity makes sense. This generalisation as well as some further applications of the technique exhibited here will be discussed in detail in some other place.

We want also to draw attention to a natural combinatorial problem arising from our considerations. Namely, it appears to be not hopeless to find an expression for (or, at least, an estimation of) the number of congruences of the G_λ-set $W[n, m, \lambda]$ in terms of m, n, and λ, and, perhaps, the subgroup lattice of the group G_λ. As it follows from the results of the paper, this would imply some estimations of the number of varieties in the intervals like $[\mathbf{Comm},\mathbf{C}_n]$.

Acknowledgement. The author is indebted to Dr. Peter Higgins for the kind invitation to participate in the Essex Conference on Transformation Semigroups and their Applications and to submit a paper in the Conference Proceedings. The success of both the Conference and the Proceedings was based entirely on Dr. Higgins' enthusiasm.

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