CONDITIONAL EQUATIONS FOR PSEUDOVARIETIES

M. V. Volkov*
Department of Mathematics and Mechanics
Ural State University
620083 Ekaterinburg, RUSSIA

Dedicated to Professor John Rhodes
on the occasion of his 60th birthday

ABSTRACT

I suggest a new language for a syntactic description of pseudovarieties that is quite close to their well-known description via pseudoidentities but has the advantage that, in this new language, every pseudovariety admits a finite basis.

Introduction

A *pseudovariety* is a class of finite algebras that is closed under the formation of homomorphic images, subalgebras, and finitary direct products. Semigroup and monoid pseudovarieties are known to coordinatize classes of recognizable languages via a correspondence discovered by Eilenberg [10]. On the other hand, the notion of a pseudovariety is obviously related to that of a variety, one of the central ideas within modern abstract algebra. Thus, through pseudovarieties, we may consequently apply various well developed algebraic approaches for studying formal languages and finite automata. A recent book by Almeida [2] gives a comprehensive account of impressive achievements gained this way.

The theory of varieties started with the famous HSP-theorem by Birkhoff establishing that structural (via closure operators) and syntactic (via identities) definitions of a variety are equivalent. I have defined pseudovarieties structurally; however, they admit a syntactic definition as well. In the literature, there exist two (different but, in a sense, equivalent, see [15]) ways to characterize pseudovarieties by means of certain equations. The first approach, found by Eilenberg and Schützenberger [11], deals with sequences (or, more generally, filters) of usual identities; this approach, however, uses a slightly different from the standard definition of what it means for such a sequence to be satisfied in a finite algebra. The second way, due to Reiterman [19] (for algebras of finite type) and Banashewski [8] (for the general case), is based on an extension of the notion of an identity to that of a pseudoidentity while the meaning of being

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satisfied is closer to the usual one. The latter approach has an important advantage since it allows to speak about finite pseudoidentity bases of a pseudovariety, a feature which has been successfully used for solving the membership problem for certain pseudovarieties.

The aim of the present paper is to introduce another syntactical description for pseudovarieties in a language generalizing that of pseudoidentities. In looking for such a new description I was motivated by the recently discovered fact that the language of pseudoidentities, though nice and useful, is not powerful enough to adequately capture the phenomenon of the polynomial decidability. I discuss this in some detail in Section 1. In Section 2 I define the crucial notions of an implicit relation and of a conditional pseudoidentity and prove a Birkhoff-type theorem. In Section 3 I demonstrate that the well-known notion of a pointlike set may be treated as a partial case of the notion of an implicit relation and deduce from that the second main result of the paper: every pseudovariety can be defined by a single conditional pseudoidentity in two variables. Section 4 contains various examples and counterexamples.

1. Preliminaries and motivation

1.1. Pseudovarieties and pseudoidentities

I briefly recall the notion of a pseudoidentity restricting for simplicity to the case of finite semigroups (the details may be found in [2, Section 3.5]). An *n-ary pseudoword* (or *implicit operation*) on a pseudovariety $\mathcal{V}$ is a $\mathcal{V}$-indexed family of functions $\pi = (\pi_S)_{S\in\mathcal{V}}$ where each $\pi_S : S^n \to S$ is an n-ary operation on S such that, for every homomorphism $\varphi : S \to T$ with $S, T \in \mathcal{V}$, the diagram

$$\begin{array}{ccc} S^n & \xrightarrow{\pi_S} & S \\ \downarrow{\scriptstyle\varphi^n} & & \downarrow{\scriptstyle\varphi} \\ T^n & \xrightarrow{\pi_T} & T \end{array}$$

commutes, that is,

$$\varphi(\pi_S(s_1, \dots, s_n)) = \pi_T(\varphi(s_1), \dots, \varphi(s_n)) \qquad (1)$$

for all $s_1, \dots, s_n \in S$. Obviously, every usual semigroup word $w(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}^+$ can be treated as a pseudoword $(w_S)_{S\in\mathcal{V}}$ where, in each finite semigroup $S \in \mathcal{V}$, the value $w_S(s_1, \dots, s_n)$ of the n-ary operation w_S is merely the value of w in S under evaluation $x_i \mapsto s_i$, $i = 1, \dots, n$. The equality (1) then holds by the very definition of a homomorphism. Thus pseudowords are indeed generalized words; in fact, they are almost words in the sense that, for every pseudoword π on $\mathcal{V}$ and for every finite semigroup $S \in \mathcal{V}$, there exists a word w such that $\pi_S(s_1, \dots, s_n) = w_S(s_1, \dots, s_n)$ for all $s_1, \dots, s_n \in S$. However, in spite of their similarity to words, pseudowords, generally speaking, need not be effectively computable. A pseudoword π on a pseudovariety $\mathcal{V}$ is said to be *computable* if there exists an algorithm which, given a finite semigroup $S \in \mathcal{V}$, computes the corresponding function π_S. If the algorithm as a function of $|S|$ requires polynomial time, π is called *polynomially computable*. An easy (but very important) example of a polynomially computable pseudoword is the unary operation $x \mapsto x^\omega$ where, in each finite semigroup S, s^ω denotes the idempotent of the subsemigroup generated by the element $s \in S$.

A *pseudoidentity* in $\mathcal{V}$ is a formal identity between pseudowords on $\mathcal{V}$, say, $\pi = \rho$, and a finite semigroup $S \in \mathcal{V}$ is said to *satisfy* this pseudoidentity if $\pi_S = \rho_S$. In particular, every usual semigroup identity $u = v$ is a pseudoidentity, and $u = v$ holds in S as an identity if and only if it does so as a pseudoidentity.

Theorem 1.1 [19]. *Let $\mathcal{V}$ be a pseudovariety of finite semigroups. A subclass $\mathcal{W} \subseteq \mathcal{V}$ is a pseudovariety if and only if there exists a set Σ of pseudoidentities in $\mathcal{V}$ such that $\mathcal{W}$ is the class of all finite semigroups in $\mathcal{V}$ satisfying all pseudoidentities from Σ.*∎

Any Σ defining $\mathcal{W}$ this way is said to be a *pseudoidentity basis* of the pseudovariety $\mathcal{W}$ within $\mathcal{V}$. If $\mathcal{W}$ admits a finite pseudoidentity basis within $\mathcal{V}$, it is called *finitely based within* $\mathcal{V}$. In the case when $\mathcal{V}$ is the pseudovariety of all finite semigroups I simply call $\mathcal{W}$ *finitely based.* The property of having a finite pseudoidentity basis is a natural and, one can say, "positive" property of pseudovarieties so it appears to be worth studying by itself. A strong additional motivation for looking for finite pseudoidentity bases arises from the interest in the decidability questions which I am going to discuss next.

1.2. The finite basis property vs. decidability

The central question about a pseudovariety is usually the decidability of its membership problem. (Recall that a pseudovariety $\mathcal{W}$ is said to have *decidable membership* if there exists an algorithm to recognize whether a given finite semigroup S belongs to $\mathcal{W}$.) Indeed, it is sufficient to remind that such famous problems as the group complexity problem for finite automata and the dot-depth problem for star-free languages can be in a natural way reformulated as the membership problems for suitable semigroup pseudovarieties. From this point of view, looking for a finite pseudoidentity basis of a pseudovariety is a promising strategy: if a pseudovariety $\mathcal{W}$ is finitely based within a decidable pseudovariety $\mathcal{V}$ and the pseudowords involved in the finite pseudoidentity basis are computable, then the membership in $\mathcal{W}$ is obviously decidable, moreover, the corresponding algorithm for testing the membership in $\mathcal{V}$ is normally rather effective. There are several striking examples when following this strategy leads to a considerable success. One of the first was a result by Almeida and Azevedo [3], see also [2, Section 9.2], solving the problem (suggested by König [17]) of identifying the least pseudovariety containing all finite $\mathscr{L}$-trivial and all finite $\mathscr{R}$-trivial semigroups. For more recent results of similar flavour see, e.g., [5, 7, 13, 18], to mention a few most important papers only. On the other hand, it turns out that, for many interesting and important pseudovarieties which have been intensively studied from the point of view of decidability, the approach via constructing a "good" finite basis fails because these pseudovarieties are proved to admit no finite basis at all, good or bad.

Consider, for example, the pseudovariety join $\mathcal{A} \vee \mathcal{G}$, that is, the least pseudovariety containing the pseudovariety $\mathcal{A}$ of all finite aperiodic semigroups and the pseudovariety $\mathcal{G}$ of all finite groups. The question (posed by Schützenberger and Rhodes, see [21, Conjecture 1.1]) of whether the membership in $\mathcal{A} \vee \mathcal{G}$ is decidable is still open, and one cannot hope to find a solution to it in the way outlined above since it was proved in [29] that $\mathcal{A} \vee \mathcal{G}$ has no finite pseudoidentity basis and even no pseudoidentity basis involving only pseudowords in finitely many variables. The same may be said about the pseudovarieties $\mathcal{O}$ and $\mathcal{PO}$ generated by all semigroups of full (respectively, partial) order-preserving transformations of a finite chain (the problem of a description of these pseudovarieties was suggested by Pin). The absence of a finite basis was proved in [20] for $\mathcal{O}$ and in [31] for $\mathcal{PO}$.

Even more confusing is, however, the circumstance that there exist decidable pseudovarieties without finite pseudoidentity basis. Recall that every pseudovariety generated by a single finite semigroup is decidable (cf. [2, Corollary 4.3.10]); on the other hand, there are (plenty of) finite semigroups generating non-finitely based pseudovarieties. The latter follows from Sapir's result [24] that if a finite semigroup S has no finite basis of (usual) identities, then the pseudovariety generated by S is not finitely based[1] and from well-known examples of finite semigroups without finite identity basis (see, e.g., [25, Chapter 2]). This type of decidable but non-finitely based pseudovarieties might appear to be not too convincing since pseudovarieties generated by a single finite semigroup play a somewhat marginal role in the general theory of pseudovarieties. There are, however, decidable but non-finitely based pseudovarieties even amongst the most important semigroup pseudovarieties. Let $\mathcal{J}$ denote the pseudovariety of all finite $\mathscr{J}$-trivial semigroups and $\mathcal{B}$ the pseudovariety of all finite bands. Zeitoun [32, 33] has shown that the pseudovariety join $\mathcal{J} \vee \mathcal{B}$ is decidable but not finitely based. Very recently Almeida, Azevedo and Zeitoun [4] and Steinberg [26] have simultaneously and independently proved that the pseudovariety join $\mathcal{J} \vee \mathcal{G}$ is also decidable while it has in shown in [27] that this pseudovariety admits no finite basis. An interesting example of a decidable but non-finitely based pseudovariety of inverse semigroups has been found by Cowan and Reilly [9].

In all these cases, the known algorithms to test the membership are quite slow (certainly superexponential) so one still might think that the absence of a finite pseudoidentity basis indeed reflects some real difficulties of the internal structure of the pseudovariety in question. However the following example of an easily decidable non-

[1] To be more precise, one should say that [24] deals with (usual) identity bases *in the class of all finite semigroups* rather than pseudoidentity bases. However, it follows from [2, Corollary 4.3.8] that these notions are equivalent in their essence for pseudovarieties generated by a single finite semigroup.

finitely based pseudovariety destroys even this last hope. Consider the pseudovariety $\mathcal{EA}$ of all finite semigroups whose idempotent generated subsemigroups are aperiodic. It is pretty easy to observe that the membership of a semigroup S in $\mathcal{EA}$ can be tested in cubic time (as a function of $n = |S|$). Indeed, given a semigroup S with n elements, we can determine the set $E(S)$ of all idempotents of S in $O(n)$ steps just by squaring each of its elements. Then let $T_0 = E(S)$ and $T_{i+1} = \{st \mid s,t \in T_i\}$; clearly,

$$T_0 \subseteq T_1 \subseteq \ldots \subseteq T_i \subseteq \ldots \tag{2}$$

and if $T_i = T_{i+1}$, then T_i is the subsemigroup T generated by $E(S)$. By the definition, $O(n^2)$ steps suffice for constructing each next T_i, and, since the chain (2) terminates after no more than n steps, we are able to construct T in $O(n^3)$ steps. Now to check whether $S \in \mathcal{EA}$, we should verify if T is aperiodic and this can be done in $O(n^2)$ steps by calculating the first n powers of each element of T. On the other hand, it was proved in [29] that $\mathcal{EA}$ has no finite pseudoidentity basis and even no pseudoidentity basis in finitely many variables. Moreover, in [30] it was shown that this pseudovariety admits no *irredundant* pseudoidentity basis (this means that, if Σ is any pseudoidentity basis of $\mathcal{EA}$ and Σ' is any finite subset of Σ, then that the set $\Sigma \setminus \Sigma'$ is still a basis of $\mathcal{EA}$). Another example of a polynomially decidable but non-finitely based pseudovariety was announced by Sapir (see [16, Theorem 3.53]); his pseudovariety is generated by a finite semigroup.

It is clear that the absence of a finite description in the pseudoidentity language for a simple pseudovariety like $\mathcal{EA}$ does not tell us that something is wrong with the pseudovariety; rather it tells us that something is wrong with the language itself. This and other examples lead to the conclusion that was already mentioned in the Introduction: unfortunately, the descriptive power of pseudoidentities is not sufficient to distinguish between decidable and undecidable or between hardly and easily decidable pseudovarieties. It provokes to search for a new, apparently more powerful language which should provide a possibility of a finite description for such finite type objects as, say, polynomially decidable pseudovarieties. The language must however be conservative enough in the sense that its propositions should define no new classes apart from usual pseudovarieties. These two requirements seem to contradict each other but it turns out to be possible to satisfy them both. The main aim of the present paper consists precisely in exhibiting a reasonable candidate for the role of such a conservative but more powerful extension of the pseudoidentity language.

2. Implicit relations and conditional pseudoidentities

2.1. Relational morphisms and implicit relations

In looking for a suitable enrichment of the language of pseudoidentities, the idea is to consider, besides pseudowords, similarly defined implicit relations. While pseudowords are families of functions commuting with homomorphisms, implicit relations are families of relations commuting with relational morphisms. Let me give a precise definition in which, for simplicity, I again restrict to the case of finite semigroups, the generalization to finite algebras of an arbitrary type being obvious.

Recall that a *relational morphism* $\mu : S \rightsquigarrow T$ of semigroups is a mapping from S into the set of non-empty subsets of T satisfying the property $\mu(s)\mu(s') \subseteq \mu(ss')$ for all $s, s' \in S$. Clearly, each homomorphism and the inverse of every surjective homomorphism are relational morphisms. The *graph* of the relational morphism $\mu : S \rightsquigarrow T$ is the set $M = \{(s,t) \in S \times T |\ t \in \mu(s)\}$ which obviously is a subsemigroup of the direct product $S \times T$. Clearly, the two projections $\varphi : M \to S$ and $\psi : M \to T$ are homomorphisms and φ is surjective so every relational morphism between S and T gives rise to a diagram of the kind

$$\begin{array}{ccc} M & \xrightarrow{\ \psi\ } & T \\ \Big\downarrow{\scriptstyle \varphi} & & \\ S & & \end{array} \tag{3}$$

Conversely, from the diagram (3), the relational morphism μ can be recovered as $\mu = \psi\varphi^{-1}$. Thus, each relational morphism is in fact a composition of the inverse of a surjective homomorphism with a homomorphism. Combining such a decomposition with the equality (1) easily yields that pseudowords commute not only with homomorphisms but also with relational morphisms in the sense that, for every pseudoword $\pi(x_1, \ldots, x_n)$ on a pseudovariety $\mathcal{V}$ and for every relational morphism $\mu : S \rightsquigarrow T$ with $S, T \in \mathcal{V}$,

$$\pi_T(t_1, \ldots, t_n) \in \mu\left(\pi_S(s_1, \ldots, s_n)\right) \tag{4}$$

whenever $t_i \in \mu(s_i)$ for all $i = 1, \ldots, n$.

If the projection $\psi : M \to T$ in (3) is injective, the relational morphism μ is called a *division* and S is said to *divide* T. Clearly, S divides T if and only if S is a homomorphic image of a subsemigroup in T.

Now let $\mathcal{V}$ be a pseudovariety of finite semigroups. An *n-ary implicit relation* on $\mathcal{V}$ is a mapping Θ associating to each semigroup $S \in \mathcal{V}$ an n-ary relation $\Theta_S \subseteq S^{(n)}$ on S such that Θ commutes with relational morphisms between members of $\mathcal{V}$ in the sense that

$$\Big(\forall\ (s_1, \ldots, s_n) \in \Theta_S\Big)\ \Big(\exists\ (t_1, \ldots, t_n) \in \Theta_T\Big) \quad t_i \in \mu(s_i) \tag{5}$$

whenever $\mu : S \rightsquigarrow T$ is a relational morphism with $S, T \in \mathcal{V}$.

Specializing this definition for the two particular kinds of relational morphisms discussed above shows that every implicit relation Θ must satisfy the following two conditions:

- for every homomorphism $\varphi : S \to T$ with $S, T \in \mathcal{V}$,

$$(s_1, \ldots, s_n) \in \Theta_S \Rightarrow (\varphi(s_1), \ldots, \varphi(s_n)) \in \Theta_T; \tag{6}$$

- for every surjective homomorphism $\varphi : S \to T$ with $S \in \mathcal{V}$

$$(t_1, \ldots, t_n) \in \Theta_T \Rightarrow \varphi^{-1}(t_1) \times \cdots \times \varphi^{-1}(t_n) \cap \Theta_S \neq \emptyset. \tag{7}$$

Conversely, since every relational morphism can be represented as a composition of the inverse of a surjective homomorphism with a homomorphism, every family $\Theta = \{\Theta_S\}_{S \in \mathcal{V}}$ of n-ary relations on semigroups of a pseudovariety $\mathcal{V}$ satisfying the conditions (6) and (7) constitutes an implicit relation on $\mathcal{V}$. This reformulation of the definition will be occasionally used in the sequel.

An n-ary implicit relation Θ on $\mathcal{V}$ is *computable* if there exists an algorithm which, given a finite semigroup $S \in \mathcal{V}$, verifies for any n-tuple $(s_1, \ldots, s_n) \in S^{(n)}$ whether $(s_1, \ldots, s_n) \in \Theta_S$. As usual, if the algorithm as a function of $|S|$ requires polynomial time, Θ is said to be *polynomially computable.*

2.2. Conditional pseudoidentities and a Birkhoff type theorem

A *conditional pseudoidentity* in a pseudovariety $\mathcal{V}$ is a pair $(\Theta, \pi = \rho)$ written as $\Theta \Rightarrow \pi = \rho$ where Θ is an n-ary implicit relation on $\mathcal{V}$ while π and ρ are n-ary pseudowords on $\mathcal{V}$. Such a pseudoidentity is said to *hold in a finite semigroup* $S \in \mathcal{V}$ whenever $\pi_S(s_1, \ldots, s_n) = \rho_S(s_1, \ldots, s_n)$ for every n-tuple $(s_1, \ldots, s_n) \in \Theta_S$. The following is an exact analogue of Theorem 1.1 for the case of conditional pseudoidentities:

Theorem 2.1. *Let $\mathcal{V}$ be a pseudovariety of finite semigroups. A subclass $\mathcal{W} \subseteq \mathcal{V}$ is a pseudovariety if and only if there exists a set Σ of conditional pseudoidentities in $\mathcal{V}$ such that $\mathcal{W}$ is the class of all finite semigroups in $\mathcal{V}$ satisfying all conditional pseudoidentities from Σ.*

Proof. The "only if" part immediately follows from Theorem 1.1 since every pseudoidentity $\pi = \rho$ may be viewed as a conditional pseudoidentity $\Theta \Rightarrow \pi = \rho$ with Θ being the universal relation of the appropriate arity on each semigroup $S \in \mathcal{V}$. To prove the "if" part I only need showing that conditional pseudoidentities are inherited by passing to divisors (homomorphic images of subsemigroups) and finitary direct products.

Thus, first let a semigroup $T \in \mathcal{V}$ satisfy a conditional pseudoidentity $\Theta \Rightarrow \pi = \rho$ and let $\mu : S \rightsquigarrow T$ be a division. Take any n-tuple $(s_1, \ldots, s_n) \in \Theta_S$; by (5) there exists an n-tuple $(t_1, \ldots, t_n) \in \Theta_T$ such that $t_i \in \mu(s_i)$ for each $i = 1, \ldots, n$. Then $\pi_T(t_1, \ldots, t_n) = \rho_T(t_1, \ldots, t_n)$ because T satisfies $\Theta \Rightarrow \pi = \rho$. Denote by M the graph of the division $\mu : S \rightsquigarrow T$ and consider the decomposition of μ into the product of the injective projection $\psi : M \to T$ with the inverse of the surjective projection $\varphi : M \to S$:

$$\begin{array}{ccc} M & \xrightarrow{\ \psi\ } & T \\ \downarrow{\scriptstyle \varphi} & & \\ S & & \end{array}$$

Let $m_i = \psi^{-1}(t_i)$, $i = 1, \ldots, n$. By (1) $\pi_T(t_1, \ldots, t_n) = \psi(\pi_M(m_1, \ldots, m_n))$ and $\rho_T(t_1, \ldots, t_n) = \psi(\rho_M(m_1, \ldots, m_n))$. Since ψ is injective, the equality

$$\pi_M(m_1, \ldots, m_n) = \rho_M(m_1, \ldots, m_n)$$

must hold. Applying the homomorphism φ to it yields

$$\pi_S(s_1, \ldots, s_n) = \rho_S(s_1, \ldots, s_n)$$

because of (1) and the fact that $\varphi(m_i) = s_i$ for $i = 1, \ldots, n$. Thus, every divisor of a semigroup with a conditional pseudoidentity keeps satisfying this pseudoidentity.

Now let $T_1, \ldots, T_k \in \mathcal{V}$ satisfy $\Theta \Rightarrow \pi = \rho$ and let $S = T_1 \times \cdots \times T_k$. Take any n-tuple $(s_1, \ldots, s_n) \in \Theta_S$ and consider the projections $\psi_j : S \to T_j$, $j = 1, \ldots, k$. Denote $\psi_j(s_i)$ by t_i^j; then $(t_1^j, \ldots, t_n^j) \in \Theta_{T_j}$ by (6) whence $\pi_{T_j}(t_1^j, \ldots, t_n^j) = \rho_{T_j}(t_1^j, \ldots, t_n^j)$ because T_j satisfies $\Theta \Rightarrow \pi = \rho$. By (1) $\pi_{T_j}(t_1^j, \ldots, t_n^j) = \psi_j(\pi_S(s_1, \ldots, s_n))$ and $\rho_{T_j}(t_1^j, \ldots, t_n^j) = \psi_j(\rho_S(s_1, \ldots, s_n))$ for all $j = 1, \ldots, m$. Since all the projections of the elements $\pi_S(s_1, \ldots, s_n)$ and $\rho_S(s_1, \ldots, s_n)$ coincide, these two elements must be equal. Thus, any direct product of semigroups with a conditional pseudoidentity preserves the pseudoidentity.■

3. Pointlike sets and a finite basis theorem

3.1. Pointlike sets as implicit relations

A subset A of a semigroup S is *pointlike with respect to a relational morphism* $\mu : S \rightsquigarrow T$ if there exists a 'point' $t \in T$ such that every element of A relates to t via μ, that is, $t \in \mu(a)$ for all $a \in A$. Given a pseudovariety $\mathcal{W}$, a subset $A \subseteq S$ is called *$\mathcal{W}$-pointlike* if A is pointlike with respect to every relational morphism between S and a semigroup in $\mathcal{W}$. I shall use the following observation which is essentially folklore and can be proved by using a standard compactness argument:

Lemma 3.1. *Let $\mathcal{W}$ be a semigroup pseudovariety, T a semigroup. There exist a semigroup $W \in \mathcal{W}$ and a relational morphism $\nu : T \rightsquigarrow W$ such that a subset of T is $\mathcal{W}$-pointlike whenever it is pointlike with respect to ν.*■

Now fix a positive integer n and a semigroup pseudovariety $\mathcal{W}$ and consider on each finite semigroup S the n-ary relation $\Lambda_S^{\mathcal{W}}$ defined as follows: an n-tuple $(s_1, \ldots, s_n) \in S^n$ belongs to $\Lambda_S^{\mathcal{W}}$ if and only if the set $\{s_1, \ldots, s_n\}$ is $\mathcal{W}$-pointlike. Let $\Lambda^{\mathcal{W}}$ be the mapping that associates to each finite semigroup S the relation $\Lambda_S^{\mathcal{W}}$. I shall call $\Lambda^{\mathcal{W}}$ *the n-ary pointlikeness relation with respect to $\mathcal{W}$*.

Proposition 3.2. *For any positive integer n and for any semigroup pseudovariety $\mathcal{W}$, the n-ary pointlikeness relation with respect to $\mathcal{W}$ is an implicit relation on the class of all finite semigroups.*

Proof. Let $\mu : S \rightsquigarrow T$ be a relational morphism between finite semigroups S and T and let a set $\{s_1, \ldots, s_n\} \subseteq S$ be $\mathcal{W}$-pointlike. Consider the semigroup $W \in \mathcal{W}$ and the relational morphism $\nu : T \rightsquigarrow W$ that controls the $\mathcal{W}$-pointlike sets in T according to Lemma 3.1. Then every element s_i, $i = 1, \ldots, n$, relates to a point $w \in W$ via the relational morphism $\nu\mu : S \rightsquigarrow W$, that is, w belongs to the set $\nu\mu(s_i)$ for each i. Since

$$\nu\mu(s_i) = \nu(\mu(s_i)) = \bigcup_{t \in \mu(s_i)} \nu(t),$$

one can choose an element $t_i \in \mu(s_i)$ such that w belongs to the set $\nu(t_i)$. Then the set $\{t_1, \ldots, t_n\} \subseteq T$ is $\mathcal{W}$-pointlike. Thus, for every n-tuple $(s_1, \ldots, s_n) \in \Lambda_S^{\mathcal{W}}$, there exists an n-tuple $(t_1, \ldots, t_n) \in \Lambda_T^{\mathcal{W}}$ such that $t_i \in \mu(s_i)$ for each $i = 1, \ldots, n$. This means that the pointlikeness relation $\Lambda^{\mathcal{W}}$ satisfies the condition (5) from the definition of an implicit relation.∎

One may view Proposition 3.2 as a source of examples of implicit relations but it should be mentioned that only for few pseudovarieties $\mathcal{W}$ the pointlikeness relation with respect to $\mathcal{W}$ is known to be computable. In fact, computing pointlikes is usually a highly non-trivial task, and such results as the decidability of pointlikes for the pseudovarieties $\mathcal{A}$ (Henckell [14]) and $\mathcal{G}$ (Ash [6]) belong to the most striking achievements of the finite semigroup theory. On the other hand, there are many examples of easily computable implicit relations (see Section 4 below). This makes me think that Proposition 3.2 should be rather viewed as an evidence that the notion of a pointlike set may be placed in more general context in which one might search for tame substitutes of hardly computable pointlikes. Certain useful properties of pointlikes can be indeed generalized to all implicit relations; this concerns, in particular, the "slice" properties recently discovered by Steinberg [26]. I shall discuss these generalizations and their applications elsewhere.

3.2. A finite basis theorem

The following lemma is well-known (and easy to deduce from Lemma 3.1):

Lemma 3.3. *A semigroup S belongs to a pseudovariety $\mathcal{W}$ if and only if every $\mathcal{W}$-pointlike set in S is a singleton.*∎

Of course, it suffices to say that every $\mathcal{W}$-pointlike set with ≤ 2 elements is a singleton, the property being equivalent to saying that the binary pointlikeness relation $\Lambda^{\mathcal{W}}(x, y)$ with respect to $\mathcal{W}$ evaluates at S as the equality relation. The latter requirement is nothing but the conditional pseudoidentity $\Lambda^{\mathcal{W}}(x, y) \Rightarrow x = y$. I have thus proved

Theorem 3.4. *Every semigroup pseudovariety can be defined in the class of all finite semigroups by a single conditional pseudoidentity in two variables.*∎

Clearly, this theorem sounds deeper as it really is because basically it is nothing but Lemma 3.3 expressed in the language of conditional pseudoidentities. However, in a certain sense, it is encouraging. In Section 4 I shall exhibit examples of effective finite bases of conditional pseudoidentities for decidable pseudovarieties without finite basis of usual pseudoidentities.

4. Examples and counterexamples

4.1. Some unary implicit relations

Specializing the general definition of an implicit relation to the unary case one gets that a unary implicit relation on a pseudovariety $\mathcal{V}$ is a mapping Θ associating to each semigroup $S \in \mathcal{V}$ a subset $\Theta(S) \subseteq S$ of S such that, for every relational morphism $\mu : S \rightsquigarrow T$ with $T \in \mathcal{V}$,

$$\Big(\forall\ s \in \Theta(S)\Big)\ \Theta(T) \cap \mu(s) \neq \emptyset. \tag{8}$$

It turns out that many "standard" subsets of finite semigroups constitute implicit relations.

Let S be a finite semigroup and let $E(S)$, $Gr(S)$, $Reg(S)$ respectively denote the set of all idempotents, all group elements, all regular elements of S. Given the multiplication table of S, all these sets are obviously polynomially computable.

Example 4.1. *Each of the mappings $S \mapsto E(S)$, $S \mapsto Gr(S)$, $S \mapsto Reg(S)$ is an implicit relation on the class of all finite semigroups.*

Proof. Let $\mu : S \rightsquigarrow T$ be an arbitrary relational morphism. Then, for every idempotent $e \in E(S)$, its image $\mu(e)$ is a subsemigroup in T so $\mu(e)$ contains an idempotent. Thus, $S \mapsto E(S)$ is an implicit relation.

Now let $g \in Gr(S)$. There exists a positive integer k such that $g^{k+1} = g$. Take an arbitrary $x \in \mu(g)$, then $x^{kn} \in \mu(g^k)$ for any positive integer n since g^k is an idempotent. For some n, the element x^{kn} is an idempotent, and therefore x^{kn+1} is a group element belonging to the set $\mu(g^{k+1}) = \mu(g)$. This means that the mapping $S \mapsto Gr(S)$ is an implicit relation.

Finally, let $a \in Reg(S)$, that is, $aba = a$ for some $b \in S$. Take an arbitrary $x \in \mu(a)$ and an arbitrary $y \in \mu(b)$. There exists a positive integer k such that $(xy)^k$ is an idempotent. Then the element $z = (xy)^{2k-1}x$ belongs to the set $\mu((ab)^{2k-1}a) = \mu(a)$ and is regular in T since

$$zyz = (xy)^{2k-1}xy(xy)^{2k-1}x = (xy)^{4k-1}x = (xy)^{2k-1}x = z.$$

Thus, the mapping $S \mapsto Reg(S)$ is also an implicit relation.∎

Further examples of the same kind may be easily produced by observing that Example 4.1 deals with those subsets of a semigroup which are so to say *pseudoverbal*, that is, consist of all values of a certain pseudoword. Indeed, in every finite semigroup S, $E(S)$ is the set of all values of the pseudoword x^ω, $Gr(S)$ equals the set of all values of the pseudoword $x^\omega x$ (usually denoted by $x^{\omega+1}$), and $Reg(S)$ coincides with the set of all values of the pseudoword $(xy)^{\omega-1}x$ where $s^{\omega-1}$ denotes the (uniquely determined) group element with the property $s^{\omega-1}s = s^\omega$ in the subsemigroup generated by the element $s \in S$. Generalizing these observations, for every pseudoword $\pi(x_1, \ldots, x_n)$ on a pseudovariety $\mathcal{V}$ and for every semigroup $S \in \mathcal{V}$, let $\Pi(S)$ denote the set of all values of π in S, that is,

$$\Pi(S) = \{\pi_S(s_1, \ldots, s_n) |\ s_1, \ldots, s_n \in S\}.$$

Proposition 4.2. *Let $\mathcal{V}$ be pseudovariety and π a pseudoword on $\mathcal{V}$. Then the mapping $S \mapsto \Pi(S)$ is an implicit relation on $\mathcal{V}$ which is [polynomially] computable whenever π is.*

Proof. Let $\mu : S \rightsquigarrow T$ be an arbitrary relational morphism between semigroups in $\mathcal{V}$. Take an element $s = \pi_S(s_1, \ldots, s_n)$ in $\Pi(S)$ and, for each $i = 1, \ldots, n$, choose an arbitrary element $t_i \in \mu(s_i)$. Then the element $t = \pi_T(t_1, \ldots, t_n)$ belongs to the set $\Pi(T)$, and the property (4) guarantees that $t \in \mu(s)$. The computability statement is obvious.∎

There are, however, unary implicit relations which cannot be produced by pseudowords. To show this, I need the next observation:

Proposition 4.3. *Let $\mathcal{V}$ be pseudovariety and Θ an implicit relation on $\mathcal{V}$. For every semigroup $S \in \mathcal{V}$ define $\Theta^*(S)$ to be the subsemigroup of S generated by the set $\Theta(S)$. Then the mapping $\Theta^* : S \mapsto \Theta^*(S)$ is an implicit relation on $\mathcal{V}$ which is [polynomially] computable whenever Θ is.*

Proof. Every element $s \in \Theta^*(S)$ can be represented as a product $s = s_1 s_2 \cdots s_n$ of some elements from the set $\Theta(S)$. Consider an arbitrary relational morphism $\mu : S \rightsquigarrow T$ between semigroups in $\mathcal{V}$. Since Θ is an implicit relation on $\mathcal{V}$, there exist elements $t_1, t_2, \ldots, t_n \in \Theta(T)$ such that $t_i \in \mu(s_i)$ for all $i = 1, \ldots, n$. Then $t = t_1 t_2 \cdots t_n \in \Theta^*(T) \cap \mu(s)$. Given the set $\Theta(S)$, the subsemigroup $\Theta^*(S)$ can be constructed in cubic of $|S|$ time — apply the reasoning on p.5 to $T_0 = \Theta(S)$.∎

In particular, combining Example 4.1 and Proposition 4.3 shows that the mapping E^* which associates to each finite semigroup S its subsemigroup generated by the set of all idempotents of S is an implicit relation on the class of all finite semigroups. Clearly, one can use this implicit relation to describe the pseudovariety $\mathcal{EA}$ of all finite semigroups whose idempotent generated subsemigroups are aperiodic: a semigroup belongs to $\mathcal{EA}$ if and only if it satisfies the conditional pseudoidentity

$$E^*(x) \Rightarrow x^\omega = x^{\omega+1}. \tag{9}$$

Thus, the pseudovariety $\mathcal{EA}$ which, I recall, has no finite basis of usual pseudoidentities (see p.5) can be effectively defined by a single conditional pseudoidentity in one variable.

Now suppose that the implicit relation E^* can be obtained as the value of some pseudoword $\pi(x_1, \dots, x_n)$. Then the conditional pseudoidentity (9) would be equivalent to the usual pseudoidentity

$$(\pi(x_1, \dots, x_n))^\omega = (\pi(x_1, \dots, x_n))^{\omega+1},$$

and the pseudovariety $\mathcal{EA}$ would have a finite pseudoidentity basis, a contradiction.

I briefly mention another similar example. Recall that the *type*-II *subsemigroup* of a finite semigroup S is the collection S_{II} of all elements $s \in S$ such that, for any relational morphism of S into a finite group, the identity element of the group is related to s. The celebrated type-II conjecture by Rhodes confirmed by Ash [6] gives a polynomial algorithm to compute S_{II} from the multiplication table of S. It can be easily checked that the mapping Σ_{II} associating to each finite semigroup its type-II subsemigroup is an implicit relation on the class of all finite semigroups. This implicit relation can be used to describe the so-called Malcev product of the pseudovariety $\mathcal{A}$ of all finite aperiodic semigroups with the pseudovariety $\mathcal{G}$ of all finite groups, that is, the pseudovariety $\mathcal{A} \textcircled{m} \mathcal{G}$ generated by all finite semigroups S such that there exists a homomorphism of S into a finite group with the preimage of the identity element of the group being aperiodic. Indeed, it follows from an old result by Rhodes and Tilson [22, Theorem 3.1] that $\mathcal{A} \textcircled{m} \mathcal{G}$ can be effectively defined by the following conditional pseudoidentity similar to (9):

$$\Sigma_{\mathrm{II}}(x) \Rightarrow x^\omega = x^{\omega+1}. \tag{10}$$

On the other hand, it was shown in [29] that the pseudovariety $\mathcal{A} \textcircled{m} \mathcal{G}$ has no finite basis of usual pseudoidentities. This implies that the implicit relation Σ_{II} cannot be obtained as a value of a pseudoword.

4.2. Some binary implicit relations

Among binary relations on a semigroup, the Green relations $\mathscr{J}$, $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$ and congruences play a distinguished role so they appear to be worth considering from the point of view of the theory of implicit relations. Let me analyze the Green relations first.

Proposition 4.4. *Each of the Green relations $\mathscr{J}$, $\mathscr{R}$ and $\mathscr{L}$ is an implicit relation on the class of all finite semigroups.*

Proof. The result is basically known (although not in the present terms). Indeed, the following lemma may be found, for example, in [23].

Lemma 4.5. *Let $\varphi : S \to T$ be a surjective homomorphism of finite semigroups, and let J' be a $\mathscr{J}$-class of T. Then $\varphi^{-1}(J') = J_1 \cup \cdots \cup J_k$ is a union of $\mathscr{J}$-classes of S, and if J_i is $\leq_{\mathscr{J}}$-minimal among $J_1, \ldots, J_k$, then $\varphi(J_i) = J'$.* ■

This lemma verifies the condition (7) in the definition of an implicit relation for the relation $\mathscr{J}$, and the condition (6) holds in the obvious way.

A literal analog of Lemma 4.5 (with the same proof) holds for the relations $\mathscr{R}$ and $\mathscr{L}$ thus yielding that they are implicit relations as well. ■

On the other hand, the relation $\mathscr{H}$ fails to satisfy the condition (7) as the following example shows. Consider two semigroup presentations:

$$S = \langle g, a |\ g^3 = g,\ g^2 a = a g^2 = a,\ a^2 = aga = 0\rangle,$$
$$T = \langle h, b |\ h^3 = h,\ hb = bh,\ h^2 b = b,\ b^2 = 0\rangle.$$

It easy to calculate that the semigroup S consists of 7 elements g, g^2, a, ag, ga, gag, and 0; the 4 elements a, ag, ga, and gag form a $\mathscr{J}$-class consisting of 4 singleton $\mathscr{H}$-classes. The semigroup T has 5 elements h, h^2, b, hb, and 0; it is obviously commutative. The elements b and hb form a $\mathscr{J}$-class being at the same time an $\mathscr{H}$-class.

$g \ \ g^2$	

a	ga
ag	gag

0

The $\mathscr{J}$-structure of S.

$h \ \ h^2$

$b \ \ hb$

0

The $\mathscr{J}$-structure of T.

One can easily check that the mapping $\begin{cases} g \mapsto h \\ a \mapsto b \end{cases}$ extends to a surjective homomorphism $\varphi : S \to T$. If $\mathscr{H}$ were an implicit relation, the fact that $b \mathscr{H} hb$ in T would force some preimage of b to be $\mathscr{H}$-related in S to some preimage of hb according to the condition (7). However $\varphi^{-1}(b) = \{a, gag\}$, $\varphi^{-1}(hb) = \{ga, ag\}$, and

$$\{a, gag\} \times \{ga, ag\} \cap \mathscr{H}_S = \emptyset.$$

Thus, unlike the other Green relations, $\mathscr{H}$ is not an implicit relation on the class of all finite semigroups. This example also shows that the intersection of two implicit relations can fail to be an implicit relation.[2]

It can be shown that the restriction of the relation $\mathscr{H}$ to the set of all regular elements is an implicit relation on the class of all finite semigroups, and the same holds true for the other Green relations.

[2]In contrast, the union of any family of implicit relations can be easily seen to be again an implicit relation.

Now let me speak about congruences. Any subpseudovariety $\mathcal{W}$ of a pseudovariety $\mathcal{V}$ defines a natural family $\Omega^{\mathcal{W}}$ of *pseudoverbal* congruences on semigroups in $\mathcal{V}$: for any $S \in \mathcal{V}$, the $\mathcal{W}$-pseudoverbal congruence $\Omega_S^{\mathcal{W}}$ on S is the intersection of all congruences θ on S such that the quotient semigroup S/θ belongs to $\mathcal{W}$. It is easy to realize that the congruence family $\Omega^{\mathcal{W}}$ is computable whenever the pseudovariety $\mathcal{W}$ is decidable. Furthermore, a semigroup $S \in \mathcal{V}$ lies in $\mathcal{W}$ if and only if the $\mathcal{W}$-pseudoverbal congruence $\Omega_S^{\mathcal{W}}$ on S equals the equality relation. This means that if the family $\Omega^{\mathcal{W}}$ were an implicit relation on $\mathcal{V}$, it could yield a [computable] conditional pseudoidentity (namely, the pseudoidentity $\Omega^{\mathcal{W}} \Rightarrow x = y$) to define an arbitrary [decidable] subpseudovariety $\mathcal{W}$ in $\mathcal{V}$ thus making the whole theory be quite close to a tautology. Fortunately, this is not the case, as the following example shows.

Let $\mathcal{SB}$ be the pseudovariety of all finite semigroups whose square is a band and $\mathcal{B}$ the pseudovariety of all finite bands. I am going to show that the family $\Omega^{\mathcal{B}}$ of $\mathcal{B}$-pseudoverbal congruences fails to be an implicit relation on $\mathcal{SB}$.

The pseudovariety $\mathcal{SB}$ is defined by the identity $xy = (xy)^2$. Denote by S the free semigroup over the set $\{a, b\}$ in the variety **SB** defined by this identity. The semigroup S is finite; in fact, it consists of 20 elements — one can directly verify this but one can also use Gerhard's solution to the word problem in the variety **SB** [12]. Thus, S belongs to the pseudovariety $\mathcal{SB}$. Let T be the Rees quotient semigroup S/S^2, and let $\varphi : S \to T$ be the natural surjective homomorphism. Clearly, $T = \{a, b, 0\}$ is a zero multiplication semigroup, and therefore, the least band congruence $\Omega_T^{\mathcal{B}}$ coincides with the universal relation. In particular, $(a, b) \in \Omega_T^{\mathcal{B}}$. If $\Omega^{\mathcal{B}}$ were an implicit relation on $\mathcal{SB}$, then the condition (7) would force some preimage of a to be $\Omega_S^{\mathcal{B}}$-related to some preimage of b. However the homomorphism φ restricted to the set $\{a, b\}$ is the identity mapping so the only preimage of a is a and the only preimage of b is b. Thus, a and b should be $\Omega_S^{\mathcal{B}}$-related, and therefore, the greatest band image of S should be trivial. This is obviously wrong since the greatest band image of S is nothing but the free band over the set $\{a, b\}$.

It can be shown that families of pseudoverbal congruences do constitute implicit relations in *congruence-permutable* pseudovarieties, in particular, within the pseudovariety of all finite groups. This will be published elsewhere.

I conclude the section with an interesting example which main idea is due to Alexei Vernitskii [28]. On each finite semigroup S, the *mirror relation* $\mathscr{M}_S$ is defined by the rule: $a \mathrel{\mathscr{M}_S} b$ if and only if there exist elements $s_1, s_2, \dots, s_n \in S$ such that $a = s_1 s_2 \cdots s_n$, $b = s_n \cdots s_2 s_1$.

Proposition 4.6. *The mapping $\mathscr{M} : S \mapsto \mathscr{M}_S$ is an implicit relation on the class of all finite semigroups.*

Proof. Let $\mu : S \rightsquigarrow T$ be a relational morphism between finite semigroups S and T, and let $a \mathrel{\mathscr{M}_S} b$. Take elements $s_1, s_2, \dots, s_n \in S$ such that $a = s_1 s_2 \cdots s_n$, $b = s_n \cdots s_2 s_1$, and pick up elements $t_1, t_2, \dots, t_n \in T$ such that $t_i \in \mu(s_i)$ for each

$i = 1, \ldots, n$. Then $t_1 t_2 \cdots t_n \in \mu(a)$, $t_n \cdots t_2 t_1 \in \mu(b)$ and, obviously,

$$t_1 t_2 \cdots t_n \mathscr{M}_T \, t_n \cdots t_2 t_1. \blacksquare$$

Although it might be not obvious from the definition, the implicit relation $\mathscr{M}$ is polynomially computable. The following is a specialization of a general algorithm invented by Vernitskii to compute various natural implicit relations.

Given a semigroup S with n elements, let μ_0 be the equality relation on S, and once the relation μ_i is defined, let the relation μ_{i+1} consist of all pairs (a, b) such that $a = ca'$, $b = b'c$ with $(a', b') \in \mu_i$, $c \in S^1$. The number of operations needed to produce μ_{i+1} equals $2n \cdot |\mu_i|$ so it has the order $O(n^3)$ (since $|\mu_i| \leq |S|^2 = n^2$ for all i). The increasing chain

$$\mu_0 \subseteq \mu_1 \subseteq \ldots \subseteq \mu_i \subseteq \ldots$$

terminates after no more than n^2 steps. It is, however, clear that the mirror relation $\mathscr{M}_S$ coincides with the union of all the relations μ_i (indeed, if $a \mathscr{M}_S b$, then, for some k, $a = s_1 s_2 \cdots s_k$ and $b = s_k \cdots s_2 s_1$ whence $(a, b) \in \mu_k$). Therefore, as soon as $\mu_i = \mu_{i+1}$, μ_i in fact equals $\mathscr{M}_S$. Thus, $\mathscr{M}_S$ can be computed in $O(n^5)$ steps.

The above observation allows to construct an effectively computable finite basis of conditional pseudoidentities for a pseudovariety which polynomial decidability was an open question so far. I mean the pseudovariety of all finite semigroups satisfying the identities[3]

$$x^2 = 0, \; xyxzx = 0, \; xy_1 \cdots y_k x y_k \cdots y_1 = 0 \quad k = 2, 3, \ldots;$$

let me denote it by $\mathcal{N}$. Almeida [1, Example 4] has observed that $\mathcal{N}$ cannot be defined by a finite set of pseudoidentities and asked whether the membership in $\mathcal{N}$ is decidable in polynomial time. This question has been repeated in [2] as Problem 5. Now it is clear that the pseudovariety $\mathcal{N}$ is defined by the conditional pseudoidentity

$$a \mathscr{M} b \Rightarrow xaxb = 0$$

together with the identities $x^2 = 0$ and $xyxzx = 0$. Since the implicit relation $\mathscr{M}$ is polynomially computable, this condition can be checked in polynomial time, and thus the membership in $\mathcal{N}$ is polynomially decidable.

It should be clear that, in a similar manner, one can construct effectively computable finite bases of conditional pseudoidentities for many pseudovarieties defined by an infinite "recurrent" series of identities or pseudoidentities.

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[3]The expression $w = 0$ is to be understood as an abbreviation for two equalities $wt = w$ and $tw = w$ where t is a free variable which does not appear in the word w.

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