The Finite Basis Problem for Finite Semigroups

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Abstract

We provide an overview of recent research on the natural question what makes a finite semigroup have finite or infinite identity basis. An emphasis is placed on results published since 1985 when the previous comprehensive survey of the area had appeared. We also formulate several open problems.

This is an updated version of July 2014 of the original survey published in 2001 in Sci. Math. Jap. **53**, no.1, 171–199. The updates (marked red in the text) mainly concern problems that were solved meanwhile and new publications in the area. The original text (in black) has not been changed.

Introduction

In his Ph.D. thesis [1966] Perkins proved that the 6-element Brandt monoid B_2^1 formed by the 2×2-matrix units

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

together with the zero and the identity 2×2-matrices, admits no finite set of laws to axiomatize all identities holding in it [see also Perkins, 1969]. His striking discovery strongly contrasted with another fundamental achievement of the equational theory of finite algebras which had appeared shortly before: we mean Oates and Powell's theorem [1964] that the identities of each finite group are finitely axiomatizable. It was this contrast that gave rise to numerous investigations whose final aim was to classify all finite semigroups with respect to the property of having/having no finite identity basis. Even though those investigations have not yet led to a solution to this major problem, they have resulted in extremely interesting and often surprising developments.

From the points of view of both the intensity and the depth of investigations, a definite peak was reached in the mid-80s. The achievements of that period were cumulated in the survey paper [Shevrin and Volkov, 1985][1]; many of them had been first announced in that survey and only then appeared in journals in a full form. There are some indications of a new peak that we are approaching at the moment due to contributions of the next generation of researchers. Therefore the time seems to be appropriate for another attempt to survey the area, to say nothing of the millennium edge which naturally provokes one to compile an account of what has been already achieved and what is still to be done.

The present paper is however not a mere continuation of Survey-85. First of all, it is less ambitious concentrating entirely on the finite basis problem for finite semigroups, while Survey-85 intended to cover the whole area "Identities of semigroups". Further, since the English version of Survey-85 is not easily accessible and the quality of the translation is rather bad, we have decided to make the present survey, to a reasonable extent, self-contained even though this has caused a few overlaps with Survey-85.

The paper is structured as follows. Section 1 gives an overview of necessary prerequisites. In Section 2 we recall the main open problems of the area. In order to create a feeling as to why the problems are so difficult to handle, we collect in Section 3 a few facts that demonstrate the extremely irregular behaviour of the class of finite semigroups with a finite identity basis with respect to almost all standard constructions and operators of semigroup theory. This section is based on Survey-85, Section 11, but we provide several references which were not available when Survey-85 appeared. The core of the paper is Section 4. There we analyze the methods developed for finding finite semigroups without a finite identity basis. It is the subarea that advanced most over the last couple of years. We give a classification of the methods and then present several recent results (of which some are not yet published). Section 5 is devoted to the opposite question: how to prove that a given finite semigroup has a finite basis. Finally, in Section 6 we list a few series of finite semigroups for which the finite basis problem appears to be of importance for further developments but resists the methods known so far.

A preliminary version of this survey has been presented in the author's lecture at the International Conference on Semigroups held in Braga in June 1999. In an expanded form, the lecture has appeared in the paper [Volkov, 2000] which contains also a few new results with full proofs. These proofs have been excluded from the present version of the survey in order to free space for a considerable amount of material not covered in [Volkov, 2000].

[1] In what follows we shall refer to this paper simply as to Survey-85.

1 Preliminaries

As far as semigroups are concerned, we adopt the standard terminology and notation from [Clifford and Preston, 1961,1967; Lallement, 1979; Howie, 1995]. Our main source for universal algebra notions is [Burris and Sankappanavar, 1981]. We recall some of those notions adapting them to the semigroup environment.

Let A be a countably infinite set called an *alphabet*. We will assume that A contains the letters x, y, z with and without indices. As usual, we denote by A^+ the *free semigroup* over A, that is, the set of all words over the alphabet A with word concatenation as the multiplication operation. Sometimes it is convenient to adjoin the empty word 1 to A^+ thus obtaining the *free monoid* A^*. By $\equiv$ we denote the equality relation on A^*.

A *non-trivial semigroup identity* over A is merely a 2-element subset $\{u, v\} \subset A^+$ usually written as $u = v$. A semigroup S *satisfies* the identity $u = v$ if the equality $u\varphi = v\varphi$ holds in S under all possible homomorphisms $\varphi : A^+ \to S$. Given S, we denote by $\mathsf{Id}\, S$ the set of all non-trivial semigroup identities it satisfies.

Given any collection Σ of non-trivial semigroup identities (an *identity system*, for short), we say that a non-trivial identity $u = v$ *follows* from Σ or is a *consequence* of Σ if every semigroup satisfying all identities of Σ satisfies the identity $u = v$ as well. The following well-known *completeness theorem of equational logic* first discovered in Birkhoff's pioneering paper [1935] provides a syntactic counterpart to this important notion:

Proposition 1.1. *A non-trivial semigroup identity $u = v$ follows from an identity system Σ if and only if there exist $w_0, w_1, \ldots, w_k \in A^+$ such that $u \equiv w_0$, $v \equiv w_k$ and, for every $i = 0, 1, \ldots, k-1$, there are $a_i, b_i \in A^*$, $s_i, t_i \in A^+$ and an endomorphism $\zeta_i : A^+ \to A^+$ such that $w_i \equiv a_i(s_i\zeta_i)b_i$, $w_{i+1} \equiv a_i(t_i\zeta_i)b_i$ and the identity $s_i = t_i$ belongs to the system Σ.*

For an identity system Σ, we denote by $\mathsf{Id}\, \Sigma$ the set of all consequences of Σ. Given a semigroup S, an *identity basis* for S is any set $\Sigma \subseteq \mathsf{Id}\, S$ such that $\mathsf{Id}\, \Sigma = \mathsf{Id}\, S$ or, in other words, such that every identity of $\mathsf{Id}\, S$ follows from Σ. A semigroup S is said to be *finitely based* if it possesses a finite identity basis; otherwise S is called *nonfinitely based*.

Let us briefly discuss an interesting subtlety which arises here. Since we are going to focus on identities of finite semigroups, it appears to be rather natural to restrict the definitions above to the class $\mathfrak{S}$ of all finite semigroups. Thus, we could say that an identity $u = v$ *follows within* $\mathfrak{S}$ from a system Σ if $u = v$ holds in every **finite** semigroup satisfying Σ, and we could then call a finite semigroup S *finitely based within* $\mathfrak{S}$ if every identity of $\mathsf{Id}\, S$ follows within $\mathfrak{S}$ from a finite subsystem $\Sigma \subseteq \mathsf{Id}\, S$. For general algebras and even for groupoids, the problem of whether or not being finitely based within the class of finite algebras is equivalent to being finitely based in the standard sense is still open. Fortunately, for semigroups the two concepts of the finite basability ("absolute" and within $\mathfrak{S}$) turn out to coincide as was shown by M. Sapir [1988]. It does not mean, however, that the "absolute" notion of a consequence of an identity system is equivalent to its "finite" counterpart! The following example by Churchill [1997] illustrates this:

Example 1.1. The identity $x^3yz^3x^2 = (yx)^3zx^3(yx)^2$ does not follow from the identity system $\Sigma = \{x^5 = y^5,\ x^3yz^3x = (yx)^3zx^3yx\}$, but follows from Σ within the class of all finite semigroups.

A similar remark can be made about the relationship between the finite basis properties of a finite monoid M as an algebra of type $\langle 2, 0\rangle$ and as a semigroup: M is finitely based within the class of all monoids if and only if it is finitely based in the standard sense, that is, in the class of all semigroups[2]. As above, this does not mean the equivalence between the "monoid" and the "semigroup" notions of a consequence of an identity system: for instance, $xy = xz$ implies $y = z$ in any monoid, but not within the class of all semigroups. Analogously, for finite semigroups with zero, treating them as algebras of type $\langle 2, 0\rangle$ does not influence the class of finitely based objects (even though it again changes the meaning of a consequence: the same identity $xy = xz$ implies the identity $xy = yz$ in any semigroup with zero).

Given a semigroup S, the class of all semigroups satisfying all identities from $\mathsf{Id}\, S$ is the *variety generated by* S; we denote this variety by $\mathsf{Var}\, S$. By the classic *HSP-theorem* by Tarski [1946], $\mathsf{Var}\, S = \mathbb{HSP}(S)$ where $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$ are respectively the operators of taking homomorphic images, subsemigroups, and direct products. We call a variety *finitely generated* if it is generated by a finite semigroup.

We will encounter also the operator $\mathbb{P}_{fin}$ of taking **finitary** direct products. Recall that a semigroup *pseudovariety* is a class of finite semigroups closed under $\mathbb{H}$, $\mathbb{S}$, and $\mathbb{P}_{fin}$. The theory of pseudovarieties has its own variant of the finite basis problem based on the notion of a *pseudoidentity*, see [Almeida, 1994]. Fortunately again, when applied to a single finite semigroup, this version of the finite basability also reduces to the standard one: a finite semigroup S possesses a finite pseudoidentity basis if and only if S has a finite identity basis [Almeida, 1989, 1994, Corollary 4.3.8].

[2]We are not sure that this claim and the next one, concerning with finite semigroups with zero, have been explicitly made in the literature, but they can be easily verified.

2 General problems

As was said in the introduction, an ultimate solution to the finite basis problem for finite semigroups would consist in a method to distinguish between finitely based and nonfinitely based finite semigroups. In more precise terms, since any finite semigroup S is an object that can be given in a constructive way (by its Cayley table, say), what we seek is an algorithm which when presented with an effective description of S, would determine whether S has a finite identity basis. This formulation of the finite basis problem as a decision problem is due to Tarski who suggested it in the early 60's in the most general setting, that is, for the class of all finite algebras [see Tarski, 1968]. We will refer to the restrictions of that general problem to various concrete classes of finite algebras (say, groupoids, semigroups, etc) as Tarski's problems for groupoids, semigroups, etc. With this convention, we may say that the research reported in the present survey groups around Tarski's problem for semigroups. Let us formulate the latter problem explicitly:

Problem 2.1. (Survey-85, Question 8.3; Sverdlovsk Notebook-89, Question 3.51) *Is there an algorithm that when given an effective description of a finite semigroup S decides if S is finitely based or not?*

Problem 2.1 is still open. We mention that, in contrast, Tarski's problem for groupoids has been recently solved in the negative by McKenzie [1996].

An algorithm is known to decide whether the **semigroup** identities of a finite inverse semigroup S possess a finite basis [Volkov, 1985]. It is based on the fact that S is finitely based if and only if the Brandt monoid B_2^1 does not belong to the variety $\mathsf{Var}\, S$ (it follows from the proof of the HSP-theorem that the latter condition can be algorithmically tested when given the Cayley table of S [see, e.g., Almeida, 1994, Section 4.3]). The "if" part was established in [Volkov, 1985], while the "only if" part is a consequence of M. Sapir's results [1987b] which we discuss in Subsection 4.4.

Here is the appropriate place for an important warning: the above algorithm does not yet provide a solution to the Tarski problem for inverse semigroups as algebras of type $\langle 2,1\rangle$. Even though it follows from a comparison between [Volkov, 1985] and [E. Kleiman, 1977] that the "inverse" (that is, of type $\langle 2,1\rangle$) identities of a finite inverse semigroup are finitely based whenever its "plain" (of type $\langle 2\rangle$) identities are, as yet we do not know whether the converse holds true. It was claimed in Survey-85, see p.19 of the English translation, that the "inverse" and the "plain" identities of every finite inverse semigroup are simultaneously finitely based. This claim, based on an announcement by M. Sapir, had spread out and had even penetrated into the handbook [Shevrin, 1991]. Later Sapir [1993] discovered that the announced result was wrong, and therefore, the question of the equivalence between the two versions of the finite basability of finite inverse semigroups should be treated as open. We note that for **infinite** inverse semigroups (in fact, even for infinite groups) the properties of being finitely based as semigroups and as algebras of type $\langle 2, 1\rangle$ are known to be independent: the group $\langle a, b \mid ab^2a = 1\rangle$ whose group identities are based by the single law $x^2y^2 = y^2x^2$ is nonfinitely based as a semigroup [Isbell, 1970; Shneerson, 1984], while the wreath product of any infinite relatively free group of exponent 4 with the countably generated free abelian group is nonfinitely based as a group [Ju. Kleiman, 1973] but satisfies no non-trivial semigroup identity [Belyaev and Sesekin, 1981] whence this product is finitely based as a semigroup.

Question 8.2 in Survey-85 asks if the algorithm from [Volkov, 1985] extends to finite orthodox semigroups. Though this question still remains open, very recently Jackson [1999, 2002] has observed that the algorithm can be used to decide whether a given finite orthodox **monoid** is finitely based. In this connection, it is also worth mentioning that every finite completely regular orthodox semigroup is finitely based [Rasin, 1982].

Kaďourek [2003b] has answered the above question in the negative. Namely, he has constructed a nonfinitely based finite orthodox semigroup S such that eSe is a band for every idempotent $e \in S$. The last property guarantees that the Brandt monoid B_2^1 does not belong to the variety $\mathsf{Var}\, S$.

Apart from purely algebraic motivations for studying finite semigroups, they are of serious interest from the point of view of formal language theory where they arise as syntactic semigroups of rational (in another terminology, regular) languages. For completeness' sake, let us briefly recall the corresponding notions referring to [Lallement, 1979, Chapter 6] for details.

A language in A^+ or A^* is merely another name for an arbitrary subset in A^+ or respectively A^*. A language in A^+ is called *rational* if it may be obtained from the singleton subsets in A and the empty set by applying a finite number of times the unary operation of generation of the subsemigroup and the binary operations of subset multiplication[3] and set-theoretical union. The definition of a rational language in A^* differs only in involving generation of the submonoid rather than the subsemigroup.

Given a language $L \subseteq A^+$, we define the relation σ_L on A^+ by

$$u \,\sigma_L\, v \quad \text{if, for any } x, y \in A^*,\ xuy \in L \Longleftrightarrow xvy \in L.$$

[3]For subsets $L, K \subseteq A^+$, their product is the set $LK = \{uv \mid u \in L, v \in K\}$.

Then σ_L turns out to be a congruence on A^+ (in fact, it is the largest congruence on A^+ for which L is a union of classes). The quotient semigroup A^+/σ_L is called the *syntactic semigroup* of the language L. In exactly the same way, one defines the *syntactic monoid* of a language in A^*.

The syntactic semigroup or monoid of a rational language L in A^+ or respectively A^* is necessarily finite; the converse is also true provided that only finitely many letters occur in words from L—this is a form (due to Myhill [1957]) of Kleene's famous theorem [1956].

In the light of this correspondence, it is rather natural to ask which combinatorial properties of a rational language ensure that its syntactic semigroup or monoid is finitely based. Almost nothing is known in this direction so far, and the question appears to be very difficult even when restricted to singleton languages (see Subsection 4.2 below). Let us formulate it as a decision problem in the flavour of Problem 2.1.

Problem 2.2. *Is there an algorithm that when given an effective description of a rational language* L (*by a rational expression, say*) *decides if the syntactic semigroup* [*or monoid*] *of* L *is finitely based or not?*

Since there exist algorithms that calculate the syntactic semigroup or monoid of a rational language from its given constructive presentation, the problem of recognizing "finitely based" rational languages in A^+ or A^* can be treated as just the Tarski problem for syntactic semigroups or respectively syntactic monoids. We note that this restricted Tarski problem seems to be quite close to the general Tarski problem for semigroups [monoids] because every finite semigroup [monoid] is a subdirect product of syntactic ones [see, e.g., Almeida, 1994, Proposition 0.3.2]. However, since neither of the properties of being finitely based/nonfinitely based is stable under forming of subdirect products (see Section 3 below), we do not yet know if the two problems are equivalent. We may only claim that a negative solution to Problem 2.2 would also answer Problem 2.1 in the negative.

Jackson [2005a] has proved that Problems 2.2 and 2.1 are equivalent; moreover, they are **polynomially** equivalent in the following sense: given a finite semigroup (or monoid) S, one can construct a finite syntactic semigroup (or monoid) S' whose size does not exceed $|S|^2$ such that S' is finitely based if and only if so is S.

From the varietal point of view, Tarski's problem is the problem of an algorithmic selection of finitely based varieties among finitely generated ones. It is also very natural to ask a "reverse" question in which one looks for an algorithm to select finitely generated varieties among finitely based ones.

This problem was also proposed by Tarski [1968], again for general algebras. It was solved in the negative by Perkins [1966, 1967] and Murskiĭ [1971]; the latter proved that even in type $\langle 2 \rangle$ there is no algorithm to determine if a given finitely based variety is generated by a finite groupoid. When further specialized to semigroups, the problem however remains open. Here is its explicit formulation:

Problem 2.3. [Survey-85, Question 8.4] *Is there an algorithm that, given a finite identity system* Σ*, decides if* $\mathsf{Id}\,\Sigma = \mathsf{Id}\,S$ *for a finite semigroup* S*?*

An algorithm is known when the system Σ contains the commutative law $xy = yx$ [O. Sapir, 1997], and even this case is far from being trivial. It would be interesting to know if this algorithm extends to the case when Σ contains a *permutation identity*, that is, an identity of the form

$$x_1 x_2 \cdots x_n = x_{1\pi} x_{2\pi} \cdots x_{n\pi} \tag{1}$$

where π is a non-identical permutation on the set $\{1, \dots, n\}$.

Recently O. Sapir [2009] has succeeded in mastering an algorithm that, given a finite identity system Σ containing a permutation identity, decides if $\mathsf{Id}\,\Sigma = \mathsf{Id}\,S$ for a finite semigroup S. The algorithm is much more involved than the one for the commutative case.

Returning to the problem of distinguishing between finitely based and nonfinitely based finite semigroups, we may ask what happens "on average" if one picks a random finite semigroup S. It turns out that such a semigroup is very likely to be finitely based. To formulate this claim in precise terms, we denote by $\mathrm{FBS}(n)$ and $\mathrm{NFBS}(n)$ the numbers of respectively finitely based and nonfinitely based semigroups with n elements. Then the ratio $\dfrac{\mathrm{NFBS}(n)}{\mathrm{FBS}(n)}$ tends to 0 as n tends to infinity. The reason for that is rather simple: it is known [see Kleitman *et al.*, 1976] that the ratio of the number of 3-nilpotent semigroups with n elements to the number of all semigroups with n elements tends to 1 as n tends to infinity, and it is easy to see that every nilpotent semigroup is finitely based. One may want to exclude the trivial case of nilpotent semigroups by switching to monoids, but it makes no real difference: if $\mathrm{FBM}(n)$ and $\mathrm{NFBM}(n)$ denote respectively the numbers of finitely based and nonfinitely based monoids with n elements, then again $\lim\limits_{n\to\infty} \dfrac{\mathrm{NFBM}(n)}{\mathrm{FBM}(n)} = 0$. The reason for that is similar to the semigroup case: on one hand, as was shown in [Koubek and Rödl, 1985], almost all monoids with n elements are of the form S^1 where S is a 3-nilpotent semigroup; on the other hand, each monoid of this form satisfies the identity $xyx = x^2y$, whence it is finitely based by a result from [Pollák and Volkov, 1985].

Since we know that both $\dfrac{\mathrm{NFBS}(n)}{\mathrm{FBS}(n)}$ and $\dfrac{\mathrm{NFBM}(n)}{\mathrm{FBM}(n)}$ are infinitesimals as n tends to infinity, the next natural step is to estimate the order of these infinitesimals. For groupoids, Murskiĭ [1979] has proved that the ratio $\dfrac{\mathrm{NFBG}(n)}{\mathrm{FBG}(n)}$ is asymptotically equal to n^{-6} (where, clearly, $\mathrm{FBG}(n)$ and $\mathrm{NFBG}(n)$ denote respectively the numbers of finitely based and nonfinitely based n-element groupoids). Having in mind an answer of a similar flavour, we formulate

Problem 2.4. [Survey-85, Question 8.5] *What is the asymptotic behaviour of the ratios* $\dfrac{\mathrm{NFBS}(n)}{\mathrm{FBS}(n)}$ *and* $\dfrac{\mathrm{NFBM}(n)}{\mathrm{FBM}(n)}$ *as n tends to infinity?*

We can also approach the finite basis problem for a random finite semigroup from a different angle. In the definition of the functions $\mathrm{FBS}(n)$ and $\mathrm{NFBS}(n)$ above, we have counted finite semigroups up to isomorphism. However, as long as we are interested in an equational property, it appears more natural to count them up to equational equivalence. (Recall that two semigroups are said to be *equationally equivalent* if they satisfy the same identities or, in other words, if they generate the same variety.) This suggests considering a new pair of functions: the numbers $\mathrm{FBV}(n)$ and $\mathrm{NFBV}(n)$ of respectively finitely based and nonfinitely based **varieties** generated by semigroups with n elements. Here there is no obvious reason to expect that an "average" finite semigroup is finitely based. For example, the mob of 3-nilpotent semigroups that previously spoiled the situation so much is not influential anymore because only 4 varieties can be generated by a 3-nilpotent semigroup. Thus, the following problem (suggested to the author by Jackson) seems to yield the correct version of the "randomized" finite basis problem:

Problem 2.5. *Evaluate* $\displaystyle\lim_{n\to\infty} \frac{\mathrm{NFBV}(n)}{\mathrm{FBV}(n)}$.

Of course, the monoid version of Problem 2.5 is of equal interest.

The last of the general problems which we want to recall is related to the notion of an irredundant identity basis. We say that an identity system Σ is *irredundant* if $\mathsf{Id}\,\Sigma' \subsetneq \mathsf{Id}\,\Sigma$ for each proper subsystem $\Sigma' \subsetneq \Sigma$. Clearly, if a semigroup S has a finite identity basis, then S also has an irredundant basis. The notion was invented soon after the first examples of nonfinitely based semigroups had arisen, in a hope that those “bad” semigroups could retain at least this property of their “good” (that is, finitely based) relatives. Unfortunately, the hope has proved to be too optimistic: not only are there various examples of finite semigroups without an irredundant identity basis [see Mashevitzky, 1983; M. Sapir, 1991; Kaďourek, 1992], but moreover no finite semigroup with an infinite irredundant basis is known so far. Now it rather appears that the answer to the following question might be negative:

Problem 2.6. (Sverdlovsk Notebook-79, Question 2.51a; Survey-85, Question 8.6) *Is there a finite semigroup with an infinite irredundant identity basis?*

Jackson [2005b] has solved Problem 2.6 in the affirmative, and moreover, he has shown that examples of finite semigroups with infinite irredundant identity bases are rather plentiful. The simplest example in Jackson [2005b] is the 9-element monoid $\mathbf{S}(\{xyxy\})$ that is discussed in some detail in Subsection 4.2 below. The problem however remains open in the monoid setting, that is, we do not yet know if there is a finite monoid with an infinite irredundant basis of **monoid** identities. All examples in Jackson [2005b] are finite monoids but it is not likely that amongst them there exists one with an infinite irredundant basis of monoid identities. Moreover, for some of those examples, it has been explicitly proved in Jackson [2005b] that they have an infinite irredundant identity basis as semigroups but no such basis as monoids.

With respect to this problem, a result by Trahtman [1987] is worth mentioning. Namely, he has proved that a 6-element semigroup possesses an infinite irredundant identity basis within a certain variety of semigroups. The 6-element semigroup that appears in this result is A_2^1, where

$$A_2 = \langle a, b \mid aba = a^2 = a,\ bab = b,\ b^2 = 0\rangle$$

is the 5-element idempotent-generated 0-simple semigroup which can be alternatively described as the semigroup formed by the following 2×2-matrices:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

The semigroup A_2 as well as the 5-element Brandt semigroup B_2 plays a distinguished role in the theory of semigroup varieties, and we will meet it again in this survey.

3 Irregularities

Let $\mathfrak{FB}$ denote the class of all finitely based finite semigroups. As we already mentioned, this class is rather irregular in the sense that—up to very few exceptions—$\mathfrak{FB}$ is closed under no standard operator or construction. In Table 1 (on the next page) we have collected a few references to results revealing such irregular behaviour. Similarly, $\mathfrak{FB}$ is not closed under taking ideals or Rees quotients, forming 0-direct unions or ordinal sums, building power semigroups, etc—cf. Survey-85, Section 11, for a detailed discussion.

Table 1: Semigroup constructions and operators vs. the finite basis property

An example of a nonfinitely based finite semigroup being:	
the *direct product* of two finitely based semigroups	can be found in [Volkov, 1989; M. Sapir, 1991; O. Sapir, 1997; Jackson, 1999];
a *subsemigroup* or a *homomorphic image* of a finitely based finite semigroup	follows from the example of a finitely based finite semigroup being the direct product of two nonfinitely based semigroups [see M. Sapir, 1991, Corollary 2.4];
the *semidirect product* or the *wreath product* of two finitely based semigroups	can be found in [Irastorza, 1985; Almeida, 1991; Tishchenko, 1996];
a *rectangular band* of finitely based semigroups	can be found in [Mashevitzky, 1984];
a *right zero band* or a *left zero band* of finitely based semigroups	follows from the previous example, see Survey-85, Section 11;
the monoid S^1 for some finitely based semigroup S	can be found in [Perkins, 1969];
a *semilattice* of finitely based semigroups or an *ideal extension* of a finitely based semigroup by another finitely based semigroup	follows from the previous example, see Survey-85, Section 11.

As for the exceptional constructions which do preserve the finite basis property, one of them is the mere adjoining 0 to a (not necessarily finite) semigroup S. The fact that the semigroup S^0 is finitely based whenever S is immediately follows from a combination of two observations. The first one is that $S \neq S^0$ implies $\mathsf{Var}\, S^0 = \mathsf{Var}\, S \vee \mathcal{SL}$, where $\mathcal{SL}$ stands for the variety of all semilattices and $\vee$ denotes the varietal join[4]; this is easy and

[4]By the *join* $\mathcal{X} \vee \mathcal{Y}$ of two semigroup varieties $\mathcal{X}$ and $\mathcal{Y}$ we mean the least variety containing both $\mathcal{X}$ and $\mathcal{Y}$; in other words, $\mathcal{X} \vee \mathcal{Y} = \mathbb{HSP}(\mathcal{X} \cup \mathcal{Y})$.

well-known. The second observation is that, given a finite identity basis for S, one can construct a finite basis for the join $\mathsf{Var}\,S \vee \mathcal{SL}$; this is due to Melnik [1970].

Another "finite basis-friendly" construction is *inflation*: an inflation T of a semigroup S is a subdirect product of S with a zero multiplication semigroup whence $\mathsf{Var}\,T = \mathsf{Var}\,S \vee \mathcal{ZM}$, where $\mathcal{ZM}$ denotes the variety of all zero multiplication semigroups. A specialization of another result by Melnik [1973] (formulated in [Melnik, 1973] in the universal algebra setting) shows how to construct a finite identity basis for the join $\mathsf{Var}\,S \vee \mathcal{ZM}$ provided a finite basis for S. Therefore T is finitely based whenever S is. As follows from [Volkov, 1984], this can be generalized in order to show that any subdirect product of a finitely based semigroup with a nilpotent semigroup is finitely based. In particular, every finite semigroup with a unique idempotent is finitely based as being a subdirect product of a group with a nilpotent semigroup [Volkov, 1984].

Lee [2014] has used the above result from [Volkov, 1984] to construct finite semigroups S such that S is nonfinitely based while the monoid S^1 is finitely based.

Since the class $\mathfrak{FB}$ fails to be $\mathbb{H}$-, $\mathbb{S}$- or $\mathbb{P}_{fin}$-closed, it is quite natural to ask for a description of the closures of $\mathfrak{FB}$ under the operators $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}_{fin}$ and their combinations. The following problem is especially intriguing:

Problem 3.1. *What is the $\mathbb{HSP}_{fin}$-closure of the class $\mathfrak{FB}$, that is, the pseudovariety generated by $\mathfrak{FB}$? In particular, is this pseudovariety finitely based?*

In connection with Problem 3.1, the following result from [Volkov, 2000] seems to be worth mentioning:

Proposition 3.1. *For each $n \geq 5$, the variety generated by all finitely based semigroups with n elements is nonfinitely based.*

4 How to prove that a finite semigroup is nonfinitely based

4.1 A classification of methods

We start with a rough description of the main ideas which underlie the known results showing the absence of a finite basis for the identities of a finite semigroup. Browsing through the literature, one may observe that in spite of the apparent diversity of the methods in use, they clearly group around four basic approaches. Here we attempt to present these approaches in the general form, while the rest of the section surveys their concrete incarnations.

1. *Syntactic analysis.* These methods directly appeal to the syntactic characterization of the deducibility of semigroup identities provided by Proposition 1.1. In order to show that $\mathsf{Id}\, S$ has no finite basis we first find a specific infinite series Σ of identities from $\mathsf{Id}\, S$ and then verify that due to the constraints caused by certain peculiarities of S, "long" identities of Σ cannot be formally deduced from any set of "short" identities in $\mathsf{Id}\, S$.

Successful implementations of this scheme include Perkins' method [1966; 1969] that brought the very first examples of nonfinitely based finite semigroups, the methods applied by Irastorza [1985] and Almeida [1991] to study the finite basis problem for certain semidirect products, Trahtman [1987] and Blanchet-Sadri's [1994] methods as well as the methods of the recent investigations by Jackson [1999, 2001], Jackson and O. Sapir [2000], O. Sapir [1997, 2000]. We briefly describe Perkins' method and its major applications and survey some results from [Jackson, 1999; Jackson and O. Sapir, 2000; O. Sapir, 1997, 2000] in Subsection 4.2.

2. *Critical semigroups.* Let $\mathcal{V} = \mathsf{Var}\, S$. For each positive integer n, we denote by $\mathcal{V}^{(n)}$ the variety defined by all identities in no more than n variables that hold in $\mathcal{V}$. Alternatively, $\mathcal{V}^{(n)}$ can be described as the class of all semigroups whose subsemigroups with no more than n generators lie in $\mathcal{V}$.

It is clear that $\mathcal{V}^{(n)} \supseteq \mathcal{V}$ for every n and that $\mathcal{V} = \mathcal{V}^{(n)}$ for some n if $\mathcal{V}$ is finitely based. On the other hand, by Birkhoff's finite basis theorem [1935], see also [Burris and Sankappanavar, 1981, §V.4], every variety $\mathcal{V}^{(n)}$ is finitely based. Thus, the equality $\mathcal{V} = \mathcal{V}^{(n)}$ for some n is not only necessary but also sufficient for S to be finitely based. Therefore showing that S is nonfinitely based is equivalent to proving that, for any n, the containment $\mathcal{V}^{(n)} \supseteq \mathcal{V}$ is strict, that is, there exists a semigroup $S_n \in \mathcal{V}^{(n)} \setminus \mathcal{V}$. We call semigroups S_n obeying the latter requirement *critical with respect to* S.

To build a series of critical semigroups, one needs a construction which is highly sensitive to removing generators. Indeed, the requirement $S_n \in \mathcal{V}^{(n)} \setminus \mathcal{V}$ says that even though S_n does not belong to the variety $\mathcal{V}$, every subsemigroup of S_n obtained by retaining only n generators does. Quite a few constructions have been invented to fulfil this requirement in various concrete situations, and it is interesting that most of them share a common feature. Namely, these constructions somehow encode a graph such that removing generators corresponds to removing edges from that graph until one arrives at its spanning tree. For instance, critical semigroups often arise as semigroups of partial transformations of a finite set or, in other words, as transition semigroups of incomplete automata, and it is the underlying graph of the corresponding automaton that stands behind the scene. As a typical example, Fig. 1 shows the transformations $\alpha_1, \dots, \alpha_n$ of the set $\{0, 1, \dots, n\}$ that generate a semigroup from the series of critical semigroups invented by Cowan and Reilly [1995] and then utilized also in [Repnitskiĭ and Volkov, 1998].

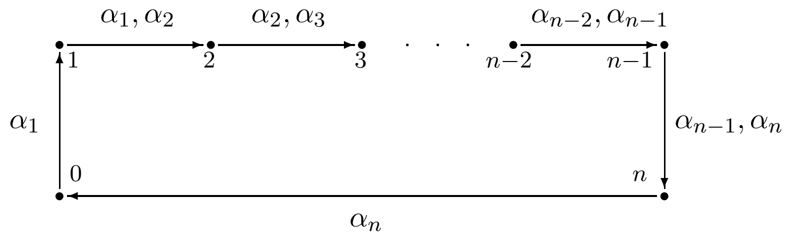


Figure 1: Generators of critical semigroup from [Cowan and Reilly, 1995]

Another tool that has proved to be quite appropriate for producing critical semigroups is the classical Rees matrix construction [see, e.g., Clifford and Preston, 1961,1967, §3.1]. For a Rees matrix semigroup $M^0(G; I, \Lambda; P)$ with the $\Lambda \times I$-sandwich-matrix $P = (p_{\lambda i})$ over a group G with zero, the "hidden" graph is the bipartite graph introduced by Houghton [1977]: it has the vertex set $I \cup \Lambda$ and the edge set $\{(i, \lambda) \in I \times \Lambda \mid p_{\lambda i} \neq 0\}$. Again we supply a simple but typical example: Fig. 2 (on the next page) shows the sandwich-matrix and the corresponding graph for the Rees matrix semigroup $M^0(\mathbb{C}_2; \mathbf{m}, \mathbf{m}; P_m)$ (where $\mathbb{C}_2 = \{e, a\}$ is the 2-element group and $\mathbf{m} = \{1, 2 \dots, m\}$). This semigroup appears in the sequence of critical semigroups used first in [Mashevitzky, 1983] and later in [Mashevitzky, 1996a; Volkov, 1996]. The reader may observe that although the critical semigroups in our two examples essentially differ as to their origin and their algebraic properties, they basically encode graphs of the same kind.

We present several methods based on the use of Rees matrix semigroups as critical semigroups in Subsection 4.3. These methods originated in a trick from aforementioned Mashevitzky's note [1983]; since then they have been applied and essentially developed in [Volkov, 1989; M. Sapir and Volkov, 1994; Mashevitzky, 1999, 2007, 2012]. "Rees matrix" methods have also been extended to the unary semigroup environment [Auinger, 1992; Auinger, Dolinka and Volkov, 2012a,b; Volkov, 1996], to pseudovarieties [Volkov, 1995, 1996], and to so-called *collective identities* of finite semigroups [Mashevitzky,

$$P_m = \begin{pmatrix} e & e & 0 & \dots & 0 & 0 \\ 0 & e & e & \dots & 0 & 0 \\ 0 & 0 & e & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & e & e \\ a & 0 & 0 & \dots & 0 & e \end{pmatrix}$$

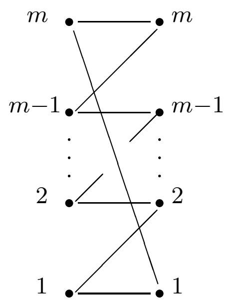


Figure 2: The matrix and the graph
of a critical semigroup from Mashevitzky [1983]

1996a].

Mastering critical semigroups as specific partial transformation semigroups was first utilized by E. Kleiman [1979] in order to show that the Brandt monoid B_2^1 is nonfinitely based as an inverse semigroup. Since the critical semigroups in his series were finite, this also gave a first proof [5] that B_2^1 is nonfinitely based in the class of finite semigroups [E. Kleiman, 1982]—the fact which cannot be extracted from Perkins' syntactic arguments. A similar approach has proved to be effective for solving the finite basis problem for several important varieties and pseudovarieties, see [E. Kleiman, 1980; Cowan and Reilly, 1995; Repnitskiĭ and Volkov, 1998; Volkov, 1998]. Very recently the methods based on partial transformation semigroups have been further developed by Cowan, Reilly, Trotter and Volkov (unpublished).

Yet another trick was used by Trotter and Volkov [1996] in the pseudovariety setting; in fact, the series of critical semigroups from this paper can be also applied in order to prove that certain finite semigroups are nonfinitely based (unpublished). For instance, the direct product $J_{15} \times G$ is nonfinitely based where G is an arbitrary non-abelian finite group and J_{15} is the 15-element $\mathcal{J}$-trivial semigroup generated by the elements $e_0, \dots, e_4$ subject to the relations

$$e_i^2 = e_i,\ e_ie_j = 0\ (i, j = 0, \dots, 4,\ j \neq i, i+1 \pmod 5),\ e_4e_0e_1 = 0.$$

3. *Finitely inexpressible properties of finitely generated varieties.* Let Θ be a property of semigroup varieties such that

1) every finitely generated variety obeys Θ;

2) any **finitely based** variety satisfying Θ must fulfil a certain additional restriction (say, possess an identity of a specific form).

[5] We note that M. Sapir's general result [1988] which we discussed in Section 1 was not yet known at that time.

Then any finite semigroup S such that $\mathsf{Var}\,S$ violates the restriction Θ is nonfinitely based. Thus, every such property Θ may be a powerful source of examples of nonfinitely based finite semigroups.

Of course, it is very far from being obvious that there exists any Θ satisfying the requirements 1) and 2) above. A striking discovery by M. Sapir [1987b] was that such a property does exist; namely, he proved that if all nil-semigroups from a finitely based variety $\mathcal{V}$ are locally finite, then $\mathcal{V}$ satisfies a non-trivial identity of the form $Z_n = w$, where the sequence $\{Z_n\}$ of *Zimin words*[6] is defined by:

$$Z_1 \equiv x_1, \text{ and } Z_{n+1} \equiv Z_n x_{n+1} Z_n.$$

Since in every finitely generated variety all semigroups are locally finite (a well-known corollary of the standard proof of the HSP-theorem), the property of varieties *to contain only locally finite nil-semigroups* satisfies both 1) and 2). Therefore if a finite semigroup S has no non-trivial identity of the form $Z_n = w$, not only is it nonfinitely based but it belongs to no locally finite finitely based variety. Semigroups obeying the latter condition are called *inherently nonfinitely based.* The inevitable Brandt monoid B_2^1 serves as a concrete example of an inherently nonfinitely based finite semigroup. We discuss the basic facts about inherently nonfinitely based finite semigroups as well as the recent developments around them in Subsection 4.4. Here, putting the notion of the inherent non-finite basability into a more general context, we would like to encourage the continued study of finitely generated varieties in the hope of discovering another finitely inexpressible property. As an example of a fairly non-obvious property of finitely generated varieties we mention the following result due to O. Sapir [1997]: *a finitely generated variety does not contain the variety of all bands.* We do not know if the latter property is finitely inexpressible.

4. *Interpretation methods.* Interpretation is a fundamental tool in the study of algorithmic problems and in complexity theory: in order to prove that a problem $\mathbf{P}$ is undecidable (hard), we usually interpret in terms of $\mathbf{P}$ another problem $\mathbf{Q}$ which is already known to be undecidable (or, respectively, hard). A similar idea may be applied to the finite basis problem which—in view of the completeness theorem of equational logic (see Proposition 1.1 above)—may be thought of as the finite axiomatizability problem

[6] Named so after Zimin whose crucial paper [1982] has revealed the role these words play in the Burnside-type problems. See Section 3.3 of the survey [Kharlampovich and M. Sapir, 1995] for an enthusiastic discussion of the history of Zimin words and of their importance.

for a specific deduction system. In order to show that the collection $\mathsf{Id}\, S$ of all identities of a given semigroup S has no finite basis, we may try to interpret within $\mathsf{Id}\, S$ another deduction system which we know is not finitely axiomatized.

Making this fuzzy idea more precise, recall that a *deduction system* is any set Q equipped with an *inference relation* $\vdash$ between Q and the set of finite subsets of Q. Let $\models$ stand for the "transitive closure" of $\vdash$: for a subset $P \subseteq Q$, $P \models q$ if and only if either $q \in P$ or there is a finite subset $R \subseteq Q$ such that $R \vdash q$ and $P \models r$ for all $r \in R$. The deduction system $(Q, \vdash)$ is said to be *finitely axiomatized* if there is a finite subset $F \subseteq Q$ such that $F \models q$ for all $q \in Q$. By a *dense interpretation* of the deduction system $(Q, \vdash)$ within $\mathsf{Id}\, S$ we mean a mapping $\bar{\cdot} : Q \to \mathsf{Id}\, S$ such that

- $\{q_1, \ldots, q_n\} \models q$ if and only if the identity $\overline{q}$ follows from the identities $\overline{q}_1, \ldots, \overline{q}_n$;
- the identity system $\overline{Q}$ forms a basis of $\mathsf{Id}\, S$.

With these definitions we immediately obtain

Proposition 4.1. *If a deduction system $(Q, \vdash)$ admits a dense interpretation within the set $\mathsf{Id}\, S$ of all identities of a semigroup S, then S is finitely based if and only if the system $(Q, \vdash)$ is finitely axiomatized.*

Thus, to show that S is nonfinitely based, it indeed suffices to densely interpret within $\mathsf{Id}\, S$ a suitable non-finitely axiomatized deduction system (or, vice versa, to densely interpret $\mathsf{Id}\, S$ within such a system).

So far the interpretation approach appears to be rather underexploited; there are however two very important applications of this idea. Mashevitzky [1984] has found an example of a nonfinitely based finite simple semigroup by interpreting the identities of a Rees matrix semigroup over a group G in the identities of G with a distinguished subset, and M. Sapir [1991] has constructed several surprising examples of nonfinitely based finite semigroups by interpreting in their identities some "weak" deduction systems defined on relatively free periodic groups. We will discuss these two interpretation methods in Subsection 4.6.

4.2 Syntactic methods

We have chosen Perkins' method to play the role of a representative of the group of syntactic methods: not only was it the very first tool developed for proving that a finite semigroup is nonfinitely based, but being simple enough, it nevertheless demonstrates two technical notions which are crucial for all such methods. The first of those notions is that of an isoterm: a word u is said to be an *isoterm relative to a semigroup* S if S satisfies no non-trivial identity of the form $u = v$; more formally, if $u \notin \bigcup \mathsf{Id}\, S$. The second one is closedness under deletion. Denote by $c(u)$ the *content* of u, that is, the set of all letters occurring in u. Suppose that a system Σ of non-trivial identities contains all its consequences. Then Σ is *closed under deletion* if for any $u = v$ in Σ, $c(u) = c(v)$ and if $|c(u)| > 1$ and all occurrences of some letter in u and in v are deleted, the resulting identity either is trivial or belongs to Σ. It is easy to see that if $S = S^1$ and S is not a group, then $\mathsf{Id}\, S$ is always closed under deletion.

Now we can formulate Perkins' result.

Theorem 4.2. [Perkins, 1969, Theorem 7] *Let S be a semigroup. Then S is nonfinitely based whenever it possesses the following four properties:*

1) *the words $xyzyx$ and $xzyxy$ are isoterms relative to S;*
2) *S satisfies neither $x^2y = yx^2$ nor $(xy)^2 = xy^2x$;*
3) *the identity system $\mathsf{Id}\, S$ is closed under deletion;*
4) *for $n = 1, 2, \dots,$ S satisfies the identity*

$$xy_1 \cdots y_n x y_n \cdots y_1 = xy_n \cdots y_1 x y_1 \cdots y_n.$$

Perkins has then verified that the Brandt monoid B_2^1 satisfies the conditions 1)–4). Another application of Theorem 4.2 in the same paper [Perkins, 1969] is the result to which we referred in Table 1: there is a finitely based finite semigroup S such that S^1 is nonfinitely based. Nowadays it is clear that the Brandt semigroup B_2 might have served as such an example since Trahtman [1981a] has proved that it is finitely based[7]. Fortunately the latter fact was not known in 1966—otherwise the following construction might not have appeared at all!

We say that a word $u \in A^*$ is a *factor* of another word $w \in A^*$ if $w \equiv vuv'$ for some $v, v' \in A^*$. Let W be a finite set of words from A^+. We denote by $\mathbf{S}(W)$ the set of all factors of words in W together with a new symbol 0 and equip this set with the following multiplication:

$$u \cdot v = \begin{cases} uv & \text{if } uv \text{ is a factor of a word in } W, \\ 0 & \text{otherwise.} \end{cases}$$

[7] Reilly [2008] has recently observed that there is a serious lacuna in the argument in [Trahtman, 1981a] and has mastered a correct proof of Trahtman's result using a (fairly nontrivial) solution to the word problem for B_2. Another, rather straightforward proof has been suggested by Lee and Volkov [2007].

More formally, $\mathbf{S}(W)$ can be defined as the Rees quotient of the free monoid A^* over the ideal

$$I(W) = \{u \in A^* \mid u \text{ is not a factor of any } w \in W\},$$

and it is easy to see that every finite Rees quotient of the free monoid is of this form.

It is clear that $\mathbf{S}(W) \setminus \{1\}$ is a nilpotent semigroup and so it is always finitely based. Perkins has observed that if $W = \{xyzyx, xzyxy, xyxy, x^2z\}$ then $\mathbf{S}(W)$ satisfies the conditions 1)–4) and thus is nonfinitely based. This gave the example he was looking for.

As a possible approach to Tarski's problem for semigroups, M. Sapir has suggested investigating the following question:

Problem 4.1. [Survey-85, Question 7.11] *Is there an algorithm that when given a finite set W of words decides if the monoid $\mathbf{S}(W)$ is finitely based or not?*

Clearly, answering Problem 4.1 in the negative will mean a negative answer to Problem 2.1 as well. We mention also a connection between a version of Problem 4.1 and Tarski's problem for syntactic monoids (Problem 2.2): it is easy to verify that if W consists of a single word, then the monoid $\mathbf{S}(W)$ is precisely the syntactic monoid of the language W. In fact, Jackson [2001] has observed that even in the case when $|W| > 1$, there quite often exists a word w such that the monoids $\mathbf{S}(W)$ and $\mathbf{S}(\{w\})$ satisfy the same identities thus being equivalent from the point of view of the finite basis property.

Though Problem 4.1 still remains open, it has motivated O. Sapir and Jackson's profound studies of the equational properties of the monoids $\mathbf{S}(W)$. First of all, they have discovered several sufficient conditions for a finite monoid S to be nonfinitely based. Each of the conditions says that S is nonfinitely based whenever certain words are isoterms relative to S, while an infinite series of words contains no isoterms relative to S.[8] Applying some of these conditions, O. Sapir [1997, 2000] has described all words w in two letters such that the monoid $\mathbf{S}(\{w\})$ is finitely based:

Theorem 4.3. *Let w be a word with $|c(w)| = 2$. The monoid $\mathbf{S}(\{w\})$ is finitely based if and only if w up to a change of letter names coincides with one of the words x^ny^m or x^nyx^n (n and m are positive integers).*

[8] The idea (of course, inspired by Theorem 4.2) to express conditions of the non-finite basability in the language of isoterms is very well suited for analyzing the finite basis problem for the monoids $\mathbf{S}(W)$ because isoterms relative to such monoids are easy to control; in particular, each factor of a word from W is an isoterm relative to $\mathbf{S}(W)$.

In particular, the monoid $\mathbf{S}(\{xyxy\})$ is nonfinitely based. It consists of 9 elements and is, as verified by Jackson [1999], the smallest nonfinitely based monoid of the form $\mathbf{S}(W)$. We note as a comparison that the monoid in the above example by Perkins has 25 elements.

In contrast with the clear description of "finitely based words" in two letters, the picture for words involving larger numbers of letters seems to be rather complicated. The following theorem combining two results by Jackson [2001] illustrates this claim.

Theorem 4.4. a) *Every word w is a factor of a word w' at most four letters longer than w so that the monoid $\mathbf{S}(\{w'\})$ is nonfinitely based. If $|c(w)| > 1$, then w' can be chosen such that $c(w') = c(w)$.*
b) *Every word w is a factor of a word w' such that the monoid $\mathbf{S}(\{w'\})$ is finitely based.*

Theorem 4.4 clearly implies

Corollary 4.5. *With every word $w \in A^+$, one can start an infinite sequence $w \equiv w_1, w_2, \dots$ of words such that w_k is a factor of w_{k+1} (so that $\mathbf{S}(\{w_k\})$ is a Rees quotient of $\mathbf{S}(\{w_{k+1}\})$) and in the sequence $\mathbf{S}(\{w_k\})$, $k = 1, 2, \dots$, finitely and nonfinitely based monoids alternate.*

One may ask if there always is a bound on the number of distinct letters in the words $w_1, w_2, \dots$. By Theorem 4.3 there exist words w in two letters (the word $xyxy$, for instance) such that, for no word w' over the set $c(w)$ having w as a factor, the monoid $\mathbf{S}(\{w'\})$ is finitely based. Jackson [2001] has constructed words in three letters (the shortest of those words has length 44) with the same property and conjectured that such words exist over any larger alphabet as well. If this is the case, any "alternating" sequence $w_1, w_2, \dots$ such as in Corollary 4.5 must include words in arbitrarily many letters.

In the general situation, that is, for monoids of the form $\mathbf{S}(W)$ with W being an arbitrary finite set of words, Jackson and O. Sapir [2000] have exhibited many strange examples showing that, in a sense, this class of monoids behaves with respect to the finite basis property as irregularly as the class of all finite semigroups. We collect two of their results in the following theorem:

Theorem 4.6. a) *With every finite set $W \subset A^+$, one can start an infinite increasing sequence $W = W_1 \subset W_2 \subset \cdots$ of finite sets of words such that in the sequence $\mathbf{S}(W_k)$, $k = 1, 2, \dots$, finitely and nonfinitely based monoids alternate.*

b) *There are finite sets $V_1, V_2 \subset A^+$ such that the monoids $\mathbf{S}(V_1)$ and $\mathbf{S}(V_2)$ are nonfinitely based [finitely based], while their direct product $\mathbf{S}(V_1) \times \mathbf{S}(V_2)$ is finitely based [respectively, nonfinitely based].*

Jackson [1999] has also shown that in some natural sense almost all monoids of the form $\mathbf{S}(W)$ are nonfinitely based—compare this with the discussion preceding the formulation of Problem 2.4. Thus, Perkins' unawareness of a finite identity basis for the Brandt semigroup B_2 has indeed given rise to a very powerful source of nonfinitely based finite semigroups!

4.3 "Rees matrix" methods

Recall that these methods use Rees matrix semigroups over a group with zero behind the scene so to speak (that is, as critical semigroups), while semigroups to which the methods apply may be of fairly general nature as in the following theorem from [Volkov, 1989]. As usual, the *core* $C(S)$ of a semigroup S is the subsemigroup of S generated by all idempotents of S.

Theorem 4.7. *Let S be a finite semigroup such that the 5-element idempotent-generated 0-simple semigroup A_2 belongs to the variety $\mathsf{Var}\, S$. If there exists a group $G \in \mathsf{Var}\, S \setminus \mathsf{Var}\, C(S)$, then S is nonfinitely based.*

Theorem 4.7 has many applications. For example, it easily implies that the semigroup T_n of all total transformations of an n-element set is nonfinitely based if $n \geq 3$. Indeed, $A_2 \in \mathsf{Var}\, T_n$ since the representation of A_2 by the right translations of either of its 3-element right ideals is faithful whence A_2 embeds into T_3. Further, the group $\mathbb{S}_n$ of all permutations of an n-element set clearly belongs to $\mathsf{Var}\, T_n$, and since all subgroups of the core $C(T_n)$ embed into $\mathbb{S}_{n-1}$, it is easy to check that $\mathbb{S}_n \notin \mathsf{Var}\, C(T_n)$ (see [Volkov, 1989] for details). Similar reasoning applies to other important types of transformation semigroups. For Rees matrix semigroups, Theorem 4.7 ensures that for any finite group G, the semigroup $M^0(G; I, \Lambda; P)$ is nonfinitely based whenever there exist $\lambda, \mu \in \Lambda$, $i, j \in I$ such that $p_{\lambda i}, p_{\lambda j}, p_{\mu j} \neq 0$, $p_{\mu i} = 0$ and the group G does not belong to the variety $\mathsf{Var}\, H$, where H is the subgroup of G generated by all non-zero entries of the sandwich-matrix P.

The unary semigroup version of Theorem 4.7 is due to Auinger, Dolinka and Volkov [2012a]. For a unary semigroup $(S, \cdot, ^*)$ we denote by $\mathrm{He}(S)$ the *Hermitian subsemigroup* of S, that is, the unary subsemigroup of S which is generated by all elements of the form xx^*. Furthermore, let C_3 be the regular $*$-semigroup $M^0(E; \mathbf{3}, \mathbf{3}; P)$ where $E = \{e\}$ is the trivial group, $\mathbf{3} = \{1, 2, 3\}$ and

$$P = \begin{pmatrix} e & e & e \\ e & e & 0 \\ e & 0 & e \end{pmatrix},$$

the unary operation * on C_3 being defined by

$$(i, e, j)^* = (j, e, i), \;\; 0^* = 0.$$

With this notation we have

Theorem 4.8. *Let S be a finite unary semigroup such that the semigroup C_3 belongs to the unary semigroup variety $\mathsf{Var}\, S$. If there exists a group $(G, \cdot, {}^{-1})$ such that $G \in \mathsf{Var}\, S \setminus \mathsf{Var}\, \mathrm{He}(S)$, then S is nonfinitely based.*

Theorem 4.8 applies to many important finite unary semigroups such as:

- the semigroup $(B_n, \circ, {}^{-1})$ of all binary relations on an n-element set, $1 < n < \infty$, endowed with the unary operation of taking the dual relation;
- the semigroup $(M_2(K), \cdot, {}^t)$ of all 2×2-matrices over a finite field K having more than two elements, endowed with transposition;
- the semigroup $(M_2(\mathbb{Z}_p), \cdot, \dagger)$ of all 2×2-matrices over the field $\mathbb{Z}_p$ where $p \equiv 3 \pmod 4$, endowed with Moore-Penrose inverse[9].

Over the last decade several types of so-called diagram monoids (Brauer monoids, partition monoids, Jones monoids, Kauffman monoids etc.) have been intensively studied in the semigroup literature. These monoids consist of elements of geometric flavor (diagrams) and have a geometrically defined multiplication as well as a natural unary operation of flipping the corresponding diagrams. Using Theorem 4.8, Auinger, Dolinka and Volkov [2012b] have shown that many of these monoids are nonfinitely based as unary semigroups.

In the recent papers [Mashevitzky, 1999, 2007] devoted to the finite basis problem for completely 0-simple semigroups, Mashevitzky has used as critical semigroups certain Rees matrix semigroups which are more complicated than those involved in the proofs of Theorems 4.7 and 4.8. It has enabled him to prove the following result:

[9] See [Drazin, 1979] for the definition and a discussion of the concept of Moore-Penrose inverse in an involution semigroup.

Theorem 4.9. *For each $m \geq 3$, the semigroup $R_m = M^0(\mathbb{C}_2; \mathbf{m}, \mathbf{m}; P_m)$, where $\mathbb{C}_2 = \{e, a\}$ is the 2-element group, $\mathbf{m} = \{1, 2\ldots, m\}$ and*

$$P_m = \begin{pmatrix} e & e & 0 & \ldots & 0 & 0 \\ 0 & e & e & \ldots & 0 & 0 \\ 0 & 0 & e & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & e & e \\ a & 0 & 0 & \ldots & 0 & e \end{pmatrix},$$

is nonfinitely based.

We note that the semigroups R_m are idempotent-generated, and therefore, Theorem 4.7 cannot be used to show that they are nonfinitely based. An interesting application of the technique from the proof of Theorem 4.9 is the result from [Mashevitzky, 2007] that the semigroup $T_2(3)$ of all nonsurjective transformations of a 3-element set is nonfinitely based.

Theorem 4.9 may seem rather special, and its proof is quite bulky. However, it is worth recalling that Mashevitzky's paper [1983] (in which the idea of using Rees matrix semigroups as critical semigroups first appeared) was also devoted to the identities of a very specific finite semigroup and these identities were studied in [Mashevitzky, 1983] via direct calculations. After a structural substitute for those calculations was found in [Volkov, 1989], the method has become flexible enough to be successfully applied in many interesting situations. Now a challenging problem is to reveal the hidden structural reasons which stay behind the calculations in [Mashevitzky, 1999, 2007], thus mastering a new powerful general condition for the non-finite basability of a finite semigroup.

4.4 Inherently nonfinitely based finite semigroups

Let us start with presenting the definition of an inherently nonfinitely based finite semigroup in a more explicit form. A nonfinitely based semigroup S is said to be *inherently nonfinitely based* if every locally finite variety $\mathcal{V}$ for which $S \in \mathcal{V}$ is also nonfinitely based[10]. M. Sapir [1987b] has proved

Theorem 4.10. *A finite semigroup S is inherently nonfinitely based if and only if all the Zimin words Z_n are isoterms relative to S.*

[10]The term "inherently nonfinitely based" was suggested by Perkins [1984], while the very first example of an inherently nonfinitely based finite algebra (in fact, a 3-element groupoid) was exhibited by Murskiĭ [1979].

M. Sapir [1987a] has given a structural description of inherently nonfinitely based finite semigroups. Recall that the *upper hypercentre* of a group G is the final term in the upper central series of that group.

Theorem 4.11. a) *A finite semigroup S is inherently nonfinitely based if and only if there exists an idempotent $f \in S$ such that the submonoid fSf is inherently nonfinitely based.*
b) *A monoid M with n elements is inherently nonfinitely based if and only if there exist $b \in M$ and an idempotent $e \in MbM$ such that if the elements ebe and $eb^{n!+1}e$ belong to the maximal subgroup H_e of M containing e, then they lie in different cosets of H_e with respect to its upper hypercentre.*

We note that Theorem 4.11 obviously yields an algorithm that when given the Cayley table of a finite semigroup S decides if S is inherently nonfinitely based or not. In contrast, it follows from McKenzie's results [1996] that no algorithm can recognize if a given finite groupoid is inherently nonfinitely based.

As mentioned in Subsection 4.1, the 6-element Brandt monoid B_2^1 is inherently nonfinitely based. Of course, this implies that every finite semigroup S such that $B_2^1 \in \mathsf{Var}\, S$ is inherently nonfinitely based as well. Moreover, Theorem 4.11 easily implies that if all subgroups of a finite semigroup S are nilpotent, then the presence of the 6-element Brandt monoid in the variety $\mathsf{Var}\, S$ is not only sufficient but also necessary for S to be inherently nonfinitely based [see M. Sapir, 1987a]. Further classes of finite semigroups whose inherently nonfinitely based members can be characterized in the same way have recently been found by Jackson [1999, 2000]:

Proposition 4.12. *If S is a finite regular semigroup with n elements, then the following are equivalent:*

(i) *S is inherently nonfinitely based;*
(ii) *$B_2^1 \in \mathsf{Var}\, S$;*
(iii) *S does not satisfy the identity $xyx = (xy)^{n!+1}x$.*

Proposition 4.13. *If the idempotents of a finite semigroup S form a subsemigroup, then S is inherently nonfinitely based if and only if $B_2^1 \in \mathsf{Var}\, S$.*

On the other hand, M. Sapir [1987a] has constructed an example of an inherently nonfinitely based finite semigroup T such that $B_2^1 \notin \mathsf{Var}\, T$. Jackson [1999, 2002] has shown that any such T must consist of at least 56 elements and contain at least 9 non-nilpotent subgroups. He has described all 56-element inherently nonfinitely based semigroups T such that $B_2^1 \notin$

$\mathsf{Var}\, T$; moreover, he has deduced from Theorem 4.11 a description of all minimal with respect to division inherently nonfinitely based finite semigroups[11]. Since an inherently nonfinitely based finite semigroup has at least one minimal inherently nonfinitely based divisor, the latter result provides another algorithmically effective characterization of inherently nonfinitely based finite semigroups.

In contrast, the following problem still remains open:

Problem 4.2. [Survey-85, Problem 9.1] *Describe all minimal* (*with respect to class inclusion*) *finitely generated inherently nonfinitely based varieties, that is, varieties $\mathcal{V}$ such that $\mathcal{V} = \mathsf{Var}\, S$ for some inherently nonfinitely based finite semigroup S, but no proper subvariety of $\mathcal{V}$ has this property.*

Even though every minimal finitely generated inherently nonfinitely based variety must be generated by a minimal inherently nonfinitely based divisor, the converse is not true: for instance, the 6-element semigroups B_2^1 and A_2^1 both are minimal inherently nonfinitely based divisors, but $\mathsf{Var}\, B_2^1 \subsetneqq \mathsf{Var}\, A_2^1$. Jackson [2002] has observed that there are infinitely many minimal finitely generated inherently nonfinitely based varieties.

We call a finite semigroup S *weakly finitely based* if S is not inherently nonfinitely based, that is, if S belongs to a locally finite finitely based variety. Clearly, the class $\mathfrak{WFB}$ of all weakly finitely based finite semigroups strictly contains the class $\mathfrak{FB}$ of all finitely based finite semigroups. From the definition, the class $\mathfrak{WFB}$ is $\mathbb{H}$- and $\mathbb{S}$-closed, and it easily follows from Theorem 4.10 that $\mathfrak{WFB}$ is also $\mathbb{P}_{fin}$-closed. Thus, $\mathfrak{WFB}$ is a pseudovariety. Using Theorem 4.11, Volkov [2000] has proved

Proposition 4.14. *The pseudovariety $\mathfrak{WFB}$ is finitely based.*

M. Sapir posed the following question:

Problem 4.3. [Survey-85, Question 11.2] *Is the pseudovariety $\mathfrak{WFB}$ generated by finitely based finite semigroups?*

Recall that in Section 3 we have formulated the problem of a description of the pseudovariety generated by all finitely based finite semigroups (Problem 3.1) to which Problem 4.3 suggests a tempting guess. In view of Proposition 4.14, in order to disprove this guess, it suffices to show that the

[11] Recall that a semigroup S is said to *divide* a semigroup T (or to be a *divisor* of T) if S is a homomorphic image of a subsemigroup of T. Clearly, the division relation when restricted to the class $\mathfrak{S}$ of all finite semigroups is a partial order.

latter pseudovariety is nonfinitely based: then it cannot coincide with the finitely based pseudovariety $\mathfrak{WFB}$.

If one focuses on the finite basis problem for finite semigroups (like we do in this survey), then the notion of an inherently nonfinitely based semigroup appears to be rather abundant. Why should we care about locally finite varieties which are not finitely generated when we are only interested in finitely generated ones? This question leads us to introduce the following notion: call a finite semigroup S *strongly nonfinitely based* if S cannot be a member of any finitely based finitely generated variety. Clearly, every inherently nonfinitely based finite semigroup is strongly nonfinitely based, and the question if the converse is true is another intriguing open problem:

Problem 4.4. *Is there a strongly nonfinitely based finite semigroup which is not inherently nonfinitely based?*

As some evidence for a positive answer to Problem 4.4 being possible, we mention the situation with a similar question for quasiidentities. Recall that a *semigroup quasiidentity* is an expression of the form

$$u_1 = v_1 \ \& \ u_2 = v_2 \ \& \ \cdots \ \& \ u_n = v_n \Longrightarrow u = v,$$

where $u_1, v_1, u_2, v_2, \dots, u_n, v_n, u, v \in A^+$. A semigroup S *satisfies* such a quasiidentity if for any homomorphism $\varphi : A^+ \to S$, $u\varphi = v\varphi$ provided that $u_1\varphi = v_1\varphi$, $u_2\varphi = v_2\varphi$, $\dots$, $u_n\varphi = v_n\varphi$. After having defined satisfaction this way, one straightforwardly proceeds with the "quasi"-analogues for all notions of the theory of identities and varieties, including those considered in this subsection. A surprising fact discovered by Margolis and M. Sapir [1995] is that no finite semigroup can be inherently nonfinitely based with respect to quasiidentities[12]. On the other hand, M. Sapir has proved in his Ph.D. thesis [1983] that there exists a finite semigroup which is strongly nonfinitely based with respect to quasiidentities (in fact, every completely simple finite semigroup which is not a rectangular group or which contains a non-abelian p-group has this property).

Let us return to the realm of identities and varieties. Here we observe that the notion of the strong non-finite basability is of special interest in the unary semigroup setting. The point is that M. Sapir [1993] has proved that no finite inverse semigroup is inherently nonfinitely based as an algebra of type $\langle 2, 1\rangle$. This result can be easilyextended to wider classes of unary semi-

[12] It is quite interesting that there exist finite **unary** semigroups which are inherently nonfinitely based with respect to quasiidentities. A method for constructing such unary semigroups (in fact, rectangular bands) has been recently developed by Lawrence and Willard [1998].

groups, for instance, to regular $*$-semigroups[13]. Having this in mind, one seeks a notion which, being weaker than the inherent non-finite basability, could be nevertheless applied to the finite basis problem for finite unary semigroups with a similar effect. The notion of a strongly nonfinitely based semigroup is a reasonable candidate here. The crucial problem is:

Problem 4.5. *Is there a finite inverse semigroup which is strongly nonfinitely based as an inverse semigroup? In particular, is the Brandt monoid B_2^1 strongly nonfinitely based as an inverse semigroup?*

The second question in this problem was first asked by E. Kleiman [1979].

Kaďourek [2003a] has almost solved Problem 4.5 in the following sense: he has proved that B_2^1 is strongly nonfinitely based with respect to the class of all inverse semigroups with solvable subgroups. In other words, every finite inverse semigroup S such that $B_2^1 \in \mathsf{Var}\, S$ is nonfinitely based provided that all subgroups of S are solvable.

4.5 A comparison between the three "standard" methods

The word "standard" in the title of this subsection is merely an abbreviation of the expression "most frequently used so far". No doubt, the three groups of methods which we presented in Subsections 4.2–4.4 above have the right to be called standard in such sense. After having discussed each of them individually, we want to compare them from the point of view of their ranges of applicability. These ranges are represented by the three boxes on Fig. 3.

We put in the boxes certain nonfinitely based finite semigroups. When a semigroup S appears in the box corresponding to one of the three standard methods, it means that the fact that S possesses no finite identity basis may be obtained by the method. Of course, if S stays in the intersection of two boxes, then each of the corresponding methods applies. The reader is already acquainted with a majority of semigroups in Fig. 3. The only new objects are the semigroups O_n and $T_n(k)$ which are semigroups of transformations of the set $\{1, 2, \ldots, n\}$: the semigroup O_n consists of all total transformations that preserve the standard order of this set, while the semigroup $T_n(k)$ $(k \leq n)$ is the ideal of the full transformation semigroup T_n consisting of all total transformations whose images have at most k elements.

The dashed line in Fig. 3 symbolizes the border between *aperiodic* finite semigroups (that is, finite semigroups having only one-element subgroups)

[13] A proof of this extension has been published by Auinger, Dolinka and Volkov [2012a].

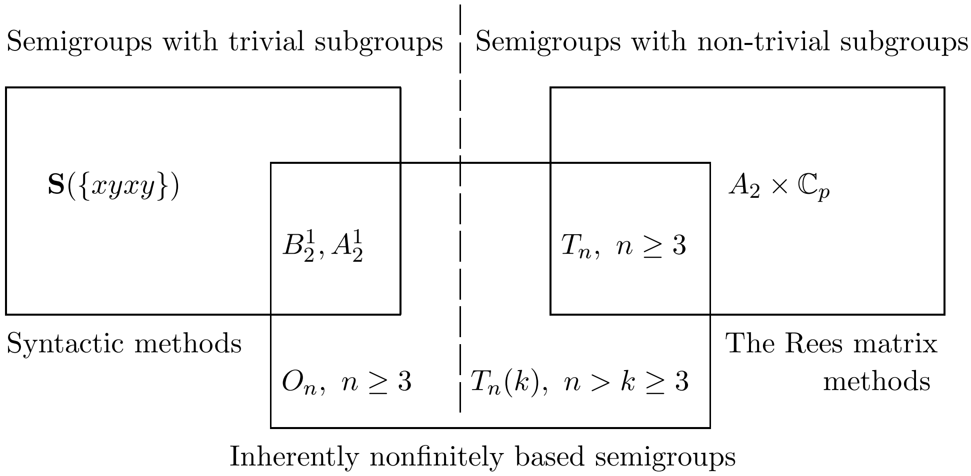


Figure 3: The ranges of applicability of the three standard methods

and all other finite semigroups. We see that so far the syntactic methods have been applied only to aperiodic finite semigroups, while any application of the Rees matrix methods has required the presence of a non-trivial subgroup. Of course, these constraints are caused by the very nature of the methods. In contrast, the method of inherently nonfinitely based semigroups can be applied to semigroups with and without non-trivial subgroups.

A further important constraint on the syntactic methods is that they only apply to monoids (or, more precisely, to semigroups sharing identities with a monoid): all these methods heavily exploit theclosure under deletion[14]. To some extent, the method of inherently nonfinitely based semigroups is also of the monoidal nature—cf. Theorem 4.11a) above—even though it can be applied to certain semigroups with no identity element (as the semigroups $T_n(k)$ with $n > k \geq 3$, for instance). The Rees matrix methods do not depend on the presence of an identity element.

In conclusion, we mention that although each of the three approaches has its own specificity, they are tightly connected with each other. For example, M. Sapir's proof [1988] that inherently nonfinitely based finite semigroups are nonfinitely based within the class of all finite semigroups uses the critical semigroup method, and the critical semigroups that appear in his proof are of the form $\mathbf{S}(\{w\})$ for suitable words $w \in A^+$.

[14] This observation is no longer true since Lee [2012] has developed a purely syntactic condition for a finite semigroup to be nonfinitely based that applies to many semigroups without identity element but does not apply to any monoid.

4.6 Interpretation methods

1. *Mashevitzky's pointed group method.* Following Bryant [1982], we call a pair (G, p), where G is a group and $p \in G$ is a fixed element considered as an additional nullary operation, a *pointed group.* A striking result by Bryant [1982] is that there exists a finite pointed group (G, p) which is nonfinitely based (as an algebra of type $\langle 2, 0 \rangle$). Mashevitzky [1984] has used Bryant's pointed group (G, p) in order to construct a nonfinitely based finite simple semigroup. In fact, Mashevitzky's semigroup is the Rees matrix semigroup over that group G with the sandwich matrix $\begin{pmatrix} e & p \\ e & e \end{pmatrix}$ where e is the identity element of G. This important example still remains the only known nonfinitely based finite completely regular semigroup; moreover, for 15 years it was the only example of a nonfinitely based variety of completely simple semigroups over a finitely based variety of groups. Only recently Auinger and Szendrei [1999] have constructed another completely simple semigroup variety with this property (their variety is not finitely generated).

The idea of relating the identities of Rees matrix semigroups with the type $\langle 2, 0, \ldots, 0 \rangle$ identities of their structure groups in which the entries of corresponding sandwich matrices play the role of distinguished constants is very natural and promising. Unfortunately, in spite of its successful debut in [Mashevitzky, 1984], it seems to have been abandoned. Mashevitzky mentioned this idea in his survey [1988] where he announced that it could be used to prove that any completely simple semigroup over a nilpotent group of finite exponent is finitely based, but no detailed proof of this result has appeared so far. There is no doubt that this interesting direction deserves more attention.

2. *Sapir's verbal subset method.* We describe this method following Survey-85, Section 11. Let S be an arbitrary semigroup, a and 0 two new symbols. Consider the semigroup $(\mathbf{T}(S), \circ)$ with the carrier set

$$S \cup \{a\} \times S^1 \cup S^1 \times \{a\} \cup \{a\} \times S^1 \times \{a\} \cup \{0\}$$

and with the multiplication extending the multiplication in S and such that for all $s_1, s_2 \in S^1$, $t \in S$,

$$(a, s_1) \circ t = (a, s_1 t), \quad t \circ (s_2, a) = (t s_2, a),$$
$$(a, s_1) \circ (s_2, a) = (a, s_1 s_2, a),$$

while all other products are equal to 0. Given a set $W \subseteq A^+$, $W(S)$ denotes the set of all values of words from W in S, that is, the union of the sets $W\varphi$

over all possible homomorphisms $\varphi : A^+ \to S$. We will call subsets of the form $W(S)$ *verbal subsets* of S. By $\mathbf{T}(S,W)$ we denote the Rees quotient of the semigroup $\mathbf{T}(S)$ over the ideal $\{a\} \times W(S) \times \{a\} \cup \{0\}$. The main property of the semigroup $\mathbf{T}(S,W)$ is revealed by the following result due to M. Sapir [cf. Survey-85, Proposition 11.1]:

Theorem 4.15. *Let F be the free semigroup over the alphabet A in the variety $\mathsf{Var}\, S$, $W \subseteq A^+$ a set of words. If $W(F) \neq V(F)$ for all finite subsets $V \subset A^+$, then the semigroup $\mathbf{T}(S,W)$ is nonfinitely based.*

The idea behind Theorem 4.15 is an interpretation of a natural deduction system $(F, \vdash)$ on the relatively free semigroup F. In this system, $P \vdash q$ (where $P \subset F$ and $q \in F$) means that there exists $p \in P$ and an endomorphism $\psi : F \to F$ such that $q = p\psi$. Clearly, subsets of F closed under $\vdash$ are precisely verbal subsets of F.

A comparison between the inference rule of $(F, \vdash)$ and that of equational logic (see Proposition 1.1) shows that, informally speaking, the inference in $(F, \vdash)$ is much weaker: in equational logic, besides using endomorphisms, we may also multiply on both sides. The construction of the semigroup $\mathbf{T}(S,W)$ is designed to "wrap" each identity of S by a new letter. This excludes using the multiplication and basically reduces the deduction apparatus to endomorphisms only, in other words, to the inference rule of $(F, \vdash)$.

Because of the "weakness" of $\vdash$, usually it is pretty easy to find a verbal subset $W(F)$ such that the subsystem $(W(F), \vdash)$ is not finitely axiomatized, that is, $W(F) \neq V(F)$ for all finite subsets $V \subset A^+$. This makes Theorem 4.15 a very powerful source of interesting examples of nonfinitely based semigroups, including finite ones. Several concrete examples found this way have been collected in M. Sapir's paper [1991]. For instance, let S be any finite non-abelian group of exponent n and denote the word $x^{n-1}y^{n-1}xy$ by $[x,y]$. If

$$W = \{[x_1,y_1],\ [x_1,y_1][x_2,y_2], \ldots, [x_1,y_1]\cdots[x_m,y_m], \ldots\},$$

then $W(F)$ is the commutator subgroup of the group F, and it is known to be not finitely generated as a verbal subset. The corresponding finite semigroup $\mathbf{T}(S,W)$ is then nonfinitely based. Taking here as S the group

$$\langle a,b,c \mid a^p = b^p = c^p = 1,\ ab = bac,\ ca = ac,\ cb = bc \rangle$$

of order p^3 and exponent p (p is an odd prime), one obtains an example of a nonfinitely based finite semigroup $T = \mathbf{T}(S,W)$ which is minimal in the following sense: the variety $\mathsf{Var}\, T$ has only finitely many subvarieties, and each proper subvariety of $\mathsf{Var}\, T$ is finitely generated and finitely based.

Again, it seems that the potential of this approach is underexploited, and it is worth looking for further applications of the method.

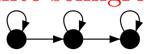
3. *Interpreting digraphs in unary Rees matrix semigroups*. By a *digraph* (directed graph) we mean a structure $G = \langle V; \rho \rangle$, where V is a set and $\rho \subseteq V \times V$ is a binary relation on V. The *adjacency semigroup* $\mathrm{A}(G)$ of G is defined on the set $(V \times V) \cup \{0\}$ and the multiplication rule is

$$(x, y)(z, t) = \begin{cases} (x, t) & \text{if } y \,\rho\, z, \\ 0 & \text{otherwise;} \end{cases}$$

$$a0 = 0a = 0 \quad \text{for all } a \in \mathrm{A}(G).$$

In terms of semigroup theory, $\mathrm{A}(G)$ is the Rees matrix semigroup over the trivial group using the adjacency matrix of the graph G as a sandwich matrix. We endow $\mathrm{A}(G)$ with an additional unary operation $a \mapsto a'$ defined as follows:

$$(x, y)' = (y, x), \quad 0' = 0.$$

Jackson and Volkov [2010] have shown that properties of digraphs expressed by so-called universal Horn sentences translate into unary identities of their adjacency semigroups. This translation preserves finite axiomatizability, and moreover, if a class of digraphs if defined by all universal Horn sentences of a finite digraph, then the corresponding variety of unary semigroups is finitely generated. This leads to a plethora of new examples of nonfinitely based unary finite semigroups. For a concrete example, consider the 3-vertex digraph H: . It is known that the universal Horn sentences holding in H are not finitely axiomatizable whence the 10-element semigroup $\mathrm{A}(H)$ is nonfinitely based as a unary semigroup.

A similar in its essence but technically more complicated method has been developed by Jackson and McKenzie [2006] in order to interpret digraphs into "plain" semigroups (with no additional operations). It also has produced a number of new examples of nonfinitely based finite semigroups interesting from the viewpoint of computational complexity.

5 How to prove that a finite semigroup is finitely based

This direction has progressed relatively slowly, and it still remains a collection of isolated theorems rather than a unified theory. Nevertheless, some of the results gathered in the subarea so far are interesting and worth discussing here.

To start with, we recall a few important classes of finitely based finite semigroups (most of them have already been mentioned above):

- finite semigroups satisfying a permutation identity (1) [Perkins, 1969, Theorem 22], in particular, commutative or nilpotent finite semigroups;
- orthodox completely regular finite semigroups [Rasin, 1982], in particular, finite groups or finite bands;
- central[15] simple finite semigroups [Survey-85, Theorem 20.3], in particular, simple finite semigroups with abelian subgroups.

Any finite semigroup S from these classes has in fact a property which is much stronger than the property of being finitely based: every variety contained in $\mathsf{Var}\, S$ is finitely based. Varieties with this property are usually called *hereditarily finitely based*; we shall apply the latter attribute to their semigroups as well.

For finite semigroups the notion of a hereditarily finitely based semigroup is in a natural sense dual to the notion of an inherently nonfinitely based semigroup: indeed, a finite semigroup S is inherently nonfinitely based if every locally finite variety containing $\mathsf{Var}\, S$ is nonfinitely based, while S is hereditarily finitely based if every [automatically, locally finite] variety contained in $\mathsf{Var}\, S$ is finitely based. By now we can only dream of a description of hereditarily finitely based finite semigroups which would be as complete as M. Sapir's description of inherently nonfinitely based finite semigroups (see Theorems 4.10 and 4.11 above). However, since the property of being hereditarily finitely based is rather strong, we do hope that the following decision problem may have a positive solution:

Problem 5.1. *Is there an algorithm that when given an effective description of a finite semigroup S decides if S is hereditarily finitely based or not?*

The corresponding problem for general algebras seems to be open. For completeness' sake, we mention a closely related decision problem, namely, the problem in which we, given a finite set Σ of identities, ask whether or not the variety defined by Σ is hereditarily finitely based. This is undecidable for groupoids [Murskiĭ, 1971] and remains open for semigroups, see Survey-85, Chapter III, for a discussion and a collection of partial results. We also mention that there is an effective description of hereditarily finitely based

[15] A completely regular semigroup S is said to be *central* if the product ef of any two idempotents $e, f \in S$ belongs to the centre of the maximal subgroup containing ef.

finite inverse semigroups (as algebras of type $\langle 2, 1\rangle$): such a semigroup is hereditarily finitely based if and only if it is a subdirect product of Brandt semigroups and/or groups [E. Kleiman, 1979].

Developing an approach suggested by Volkov and M. Sapir [1988], Jackson [2000] has recently found positive solutions to the restrictions of Problem 5.1 to certain classes of monoids:

Proposition 5.1. *Let M be a finite monoid verifying one of the conditions:*

(i) *the word x^2 is an isoterm relative to M;*
(ii) *M is isomorphic to a Rees quotient of the free monoid A^*.*

Then M is hereditarily finitely based if and only if M satisfies one of the identities $xyx = x^2y$ or $xyx = yx^2$.

Proposition 5.2. *Let M be a finite orthodox monoid. Then M is hereditarily finitely based if and only if M is completely regular.*

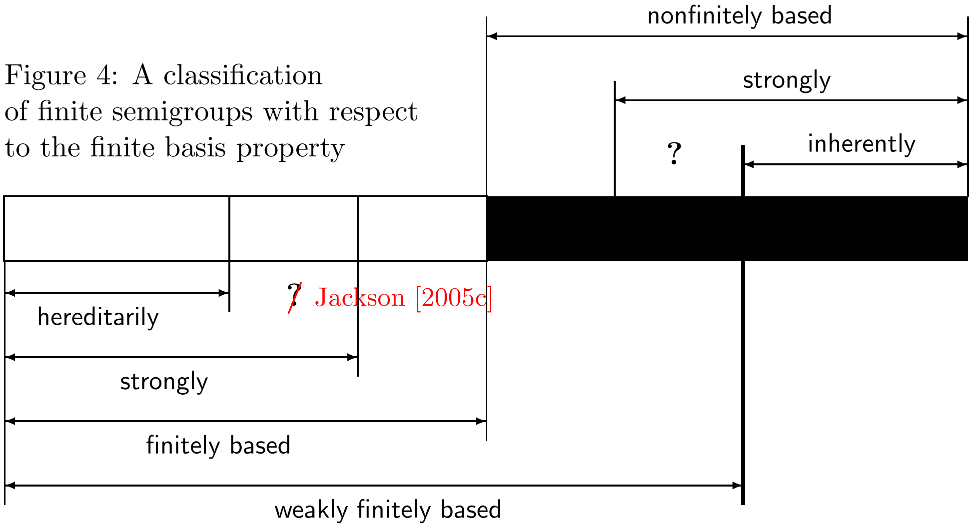
We have observed a duality between the notions of a hereditarily finitely based and an inherently nonfinitely based semigroup. In Subsection 4.4 we have argued that from the "finite" standpoint, a notion which we have called the strong nonfinite basability appears to be more natural than that of the inherent nonfinite basability. Arguments of the same type apply to the notion of a hereditarily finitely based semigroup. A finitely generated variety may contain plenty of subvarieties which are not finitely generated (even uncountably many of them as the example of the variety $\mathsf{Var}\, A_2^1$ shows [see Trahtman, 1988]). Why should we care about these subvarieties when we are only interested in finite semigroups? Thus, we call a finite semigroup S *strongly finitely based* if every finite semigroup from the variety $\mathsf{Var}\, S$ is finitely based. Clearly, every hereditarily finitely based finite semigroup is strongly finitely based, and the question of whether the converse is true constitutes a problem which is in a sense dual to Problem 4.4:

Problem 5.2. *Is there a strongly finitely based finite semigroup which is not hereditarily finitely based?*

Jackson [2005c] has solved Problem 5.2 in the affirmative. Namely, he has proved that the 7-element monoid $\mathbf{S}(\{xyx\})$ (which is finitely based by Theorem 4.3) is strongly finitely based but not hereditarily finitely based.

In order to help the reader who at this point might be already overwhelmed by too many versions of the notions of being finitely/nonfinitely based, we put them all on Fig. 4. The question marks there correspond to

Figure 4: A classification of finite semigroups with respect to the finite basis property



Problems 4.4 and 5.2, while the bold line symbolizes the only border that has been effectively localized so far.

We conclude this section with a brief overview of results establishing the finite basis property for a few specific semigroups.

Trahtman [1991] has published a proof of his theorem first announced in [Trahtman, 1983] that every 5-element semigroup is finitely based. We refer to Survey-85, Section 10, for a detailed discussion of the history of the problem and the previous steps towards its solution. Of course, [Trahtman, 1991] contains many clever tricks which may be useful for finding a finite identity basis for further classes of finite semigroups. Recently Lee [2013] has published a modified, very readable proof of Trahtman's theorem.

Mashevitzky [1994] has proved that the semigroup $M_n(1)$ of all $n \times n$-matrices of rank ≤ 1 over any field is finitely based. This completes a certain chapter in the study of the finite basis problem for semigroups of $n \times n$-matrices because the answer to this problem for semigroups $M_n(k)$ of all $n \times n$-matrices of rank $\leq k$ was already known for all $k > 1$. Namely, if the ground field is finite, then the semigroup $M_n(k)$ with $k > 1$ is nonfinitely based and even inherently nonfinitely based because it contains a subsemigroup isomorphic to the Brandt monoid B_2^1; if the ground field is infinite, then the semigroup $M_n(k)$ with $k > 1$ satisfies no non-trivial identity; this follows from an observation due to Golubchik and Mikhalev [1978].

Mashevitzky [1996b] has introduced and studied the notion of a left hereditary system of semigroup identities. Let u and w be words. By $u|_w$ we denote the longest prefix of u not containing w as a factor. For any identity $u = v$, the identity $u|_w = v|_w$ is called *the left section of* *the identity $u = v$ relative to w.* A system of identities containing all left sections of each of its identities relative to all words of length n is said to be *left n-hereditary.* Left hereditary identity systems often arise as the identity systems of semigroups that are extensions of a left zero ideal. For example, if S is a subdirectly irreducible semigroup having a non-trivial left zero ideal, then $\mathsf{Id}\, S$ is left 1-hereditary [Mashevitzky, 1996b]. Another example is the semigroup $T_n(2)$ of all transformations of an n-element set whose images have at most 2 elements. If $n \geq 5$, then the identity system $\mathsf{Id}\, T_n(2)$ has been shown by Torlopova [1982] to be left 2-hereditary.

The main results of [Mashevitzky, 1996b] are two propositions showing that, in certain situations, the condition of being left hereditary suffices to ensure that the ideal extension of a left zero semigroup by a finitely based semigroup is again finitely based.

Proposition 5.3. *Let T be a semigroup with 0 but without zero divisors satisfying the identity $x^k = x$ for some $k > 1$. Suppose that S is an ideal extension of a left zero ideal by the semigroup T and that the identity system $\mathsf{Id}\, S$ is left 1-hereditary. Then S is finitely based whenever T is.*

Proposition 5.4. *Let $T = M^0(G; I, \Lambda; P)$ be a Rees matrix semigroup over a group G of finite exponent, and there exist $\lambda, \mu \in \Lambda$, $i, j \in I$ such that $p_{\lambda i}, p_{\lambda j}, p_{\mu j} \neq 0$, $p_{\mu i} = 0$. Suppose that S is an extension of a left zero ideal by the semigroup T and that the identity system $\mathsf{Id}\, S$ is 2-hereditary. Then S is finitely based whenever T is.*

An important application of Proposition 5.4 is the fact that the semigroup $T_n(2)$ is finitely based provided that $n \geq 5$. This is a partial solution to Question 22.1 in Survey-85. Recall that the semigroup $T_3(2)$ is nonfinitely based (we mentioned this result from [Mashevitzky, 2007] in Subsection 4.3 above). Since the 4-element semigroup $T_2(2) = T_2$ is obviously finitely based, the only remaining member of the family $\{T_n(k)\}$ for which the finite basis problem is still open is the 88-element semigroup $T_4(2)$. Mashevitzky once announced (in his abstract submitted at XVIII All-Union Algebra Conference held in Kishinev in September 1985) that $T_4(2)$ is finitely based but this announcement was never confirmed in a detailed publication. Recently Mashevitzky [2012] has proved that $T_4(2)$ is nonfinitely based.

6 Concrete problems

We conclude our survey with a list of open questions concerning with the finite basis property for certain concrete finite semigroups. The first of these questions has been suggested by Rasin:

Problem 6.1. [Survey-85, Question 8.1] *Let* $\mathbb{C}_p = \langle a \mid a^p = e\rangle$ *be the cyclic group of prime order* p, $M_p = M(\mathbb{C}_p; \mathbf{2}, \mathbf{2}; P)$ *where* $\mathbf{2} = \{1, 2\}$, $P = \begin{pmatrix} e & a \\ e & e \end{pmatrix}$. *Is the semigroup* M_p^1 *finitely based?*

In order to explain the importance of Problem 6.1, we recall that the only example of nonfinitely based finite completely regular semigroup known so far is Mashevitzky's completely simple semigroup over Bryant's pointed group (we discussed it in Subsection 4.6 above). In particular, it is still unknown whether or not every finite band of abelian groups is finitely based. The semigroups M_p^1 constitute natural candidates here. Furthermore, if all semigroups M_p^1 are nonfinitely based (as we conjecture), then by a result by Rasin [1981], every hereditarily finitely based band of groups must decompose into a subdirect product of completely simple semigroups (possibly, with zero adjoined) and bands. This would constitute a major step towards a description of hereditarily finitely based finite completely regular semigroups.

Petrich [2003] has shown that the variety $\mathsf{Var}\, M_p^1$ has 32 subvarieties and has found finite identity bases for all proper subvarieties. Thus, if the semigroup M_p^1 is nonfinitely based, it generates a minimal (with respect to inclusion) nonfinitely based variety of completely regular semigroups. The existence of such varieties is an easy consequence of Zorn's lemma but no concrete example is known so far. This makes Problem 6.1 even more intriguing.

The next question deals with a natural class of semigroups of matrices:

Problem 6.2. [Survey-85, Question 22.2] *Is the semigroup of all upper triangular* $n \times n$*-matrices* $(n \geq 2)$ *over a finite field finitely based?*

A partial solution to this problem has been found by Volkov and Goldberg [2003]. Namely, they have shown that the semigroup of all upper triangular $n \times n$-matrices over a finite field K is even inherently nonfinitely based provided that $n \geq 4$ and $|K| > 2$. The case when the ground field has exactly 2 elements and the size of matrices is 2 has been recently settled by Zhang, Li and Luo [2012, 2013] who first proved that the corresponding semigroup is finitely based and then verified that it is even hereditarily finitely based. All other cases still remain open.

Volkov and Goldberg [2004] considered the semigroup of all upper triangular $n \times n$-matrices over the Boolean semiring $\langle\{0, 1\}; +, \cdot\rangle$ in which

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0 + 0 = 0, \quad 1 \cdot 1 = 1 + 0 = 0 + 1 = 1 + 1 = 1.$$

They proved that the semigroup is inherently nonfinitely based provided that $n \geq 4$. For $n = 2$ or 3, Volkov and Goldberg [2004] claimed that the semigroup is not inherently nonfinitely based but in fact their argument worked only for $n = 2$. This has been observed by Li and Luo [2011] who have verified that the semigroup of all upper triangular Boolean $n \times n$-matrices is inherently nonfinitely based if $n = 3$ and is finitely based if $n = 2$.

Next, we want to present a bunch of questions concerning with certain important semigroups of partial transformations of the set $\{1, 2, \ldots, n\}$. In order to introduce all these semigroups in a concise way, we borrow from [Volkov, 1998] the idea of putting them together in what we call the *basic frame* of partial transformation semigroups. The latter is the subsemilattice of the $\cap$-semilattice of all semigroups of partial transformations of $\{1, 2, \ldots, n\}$ generated by:

- T_n, the semigroup of all total transformations;
- I_n, the inverse symmetric semigroup, that is, the semigroup of all injective partial transformations;
- PO_n, the semigroup of order-preserving partial transformations;
- PE_n, the semigroup of all extensive partial transformations (recall that a partial transformation α is said to be *extensive* if $i.\alpha \geq i$ for every i in the domain of α).

For $n > 2$, the basic frame consists of 13 semigroups named on Fig. 5 (on the next page). Each of these semigroups is thus characterized by a combination of some of the four just mentioned fundamental properties of transformations: being total, injective, order preserving, or decreasing. For example, the semigroup $\mathbb{S}_n = T_n \cap I_n$ consists of all total and injective transformations and is, of course, nothing else but the group of all permutations of $\{1, 2, \ldots, n\}$. See the survey [Higgins, 1999] for an interesting discussion of the distinguished role the non-trivial semigroups in the basic frame play in the theory of finite semigroups.

The dotted curves on Fig. 5 separate semigroups for which a solution to the finite basis problem is known from "terra incognita" (with one small exception in the case $n = 3$: the 5-element semigroup OE_3 is, of course, finitely based). The red solid line added to the original picture shows the nowadays established border between finitely based and nonfinitely based semigroups in the basic frame for all sufficiently large n. See the comments after Problem 6.3 for more detail. The "positive" area needs no comment; as for the nonfinitely based semigroups on Fig. 5, they all are even inherently

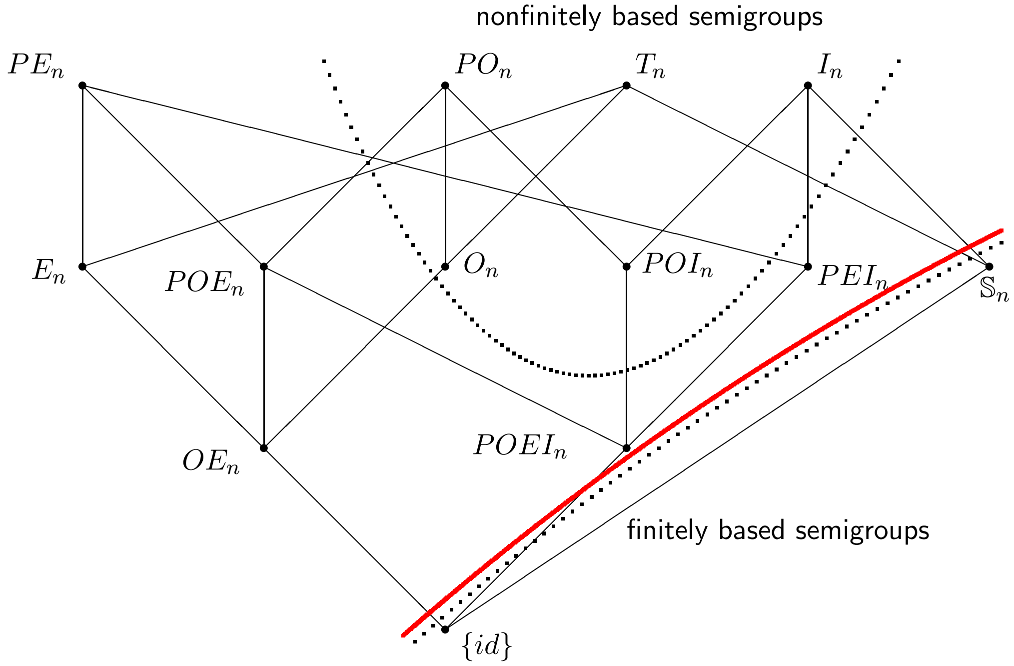


Figure 5: The basic frame of transformation semigroups ($n > 2$)

nonfinitely based because each of them generates a variety containing B_2^1 alias POI_2. Let us explicitly formulate the remaining questions:

Problem 6.3. *Which of the transformation semigroups PE_n, PEI_n, E_n, POE_n, $POEI_n$ $(n > 2)$ and OE_n $(n > 3)$ are finitely based?*

Volkov [2004] has solved the finite basis problem for the series OE_n: the semigroup OE_n is finitely based if and only if $n \leq 4$. Goldberg [2007] has proved that, for each positive integer n, the monoids E_{n+1}, PE_n and POE_n satisfy the same identities and that these monoids are nonfinitely based whenever $n \geq 4$. S. O. Ivanov announced at the 57th conference "Herzen Readings" held in St Petersburg in 2003 a finite identity basis for the 6-element monoid E_3 but didn't published his proof. A proof was then published by Lee [2009] who also proved a much stronger result: the monoid E_3 is in fact hereditarily finitely based.

We also ask the following tantalizing question:

Problem 6.4. *Is the symmetric inverse semigroup I_n $(n \geq 2)$ finitely based as an* ***inverse semigroup****?*

Recall that we have discussed the current state of art around the relationship between the "inverse" and the "plain" versions of the finite basis problem for finite semigroups in Section 2. The main result of Kaďourek [2003a] readily implies that I_2, I_3, and I_4 are nonfinitely based as inverse semigroups.

We started our story with the 6-element Brandt monoid B_2^1; let us finish it by formulating an important open question concerning the 5-element Brandt semigroup B_2:

Problem 6.5. [Jackson, 2000, Question 4.6] *Is the* 5*-element Brandt semigroup* B_2 *hereditarily finitely based?*

If the answer is "No", the structure of hereditarily finitely based finite semigroups would become much clearer. We mention that Jackson [2000] has shown that the direct product of B_2 with the 3-element monoid $\{1, a, 0\}$ in which $a^2 = 0$ is not hereditarily finitely based.

Lee [2004] has shown that the answer is "Yes". Moreover, later Lee [2008] has shown that even the 5-element semigroup A_2 is hereditarily finitely based. It is known that the variety generated by A_2 contains all completely 0-simple combinatorial semigroups, in particular, B_2.

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